Reasoning with !-Graphs

Alexander Merry
Magdalen College
University of Oxford

A thesis submitted for the degree of
Doctor of Philosophy

Trinity 2013

Acknowledgements

Firstly, I would like to thank my supervisors, Professor Samson Abramsky and Professor Bob Coecke, for giving me the opportunity to do this research and supporting me in my efforts, as well as for starting the field that was the motivation for this work. I also owe an immense debt of gratitude to Dr. Aleks Kissinger, who both paved the way for my work and gave me invaluable advice, encouragement and feedback on the way.

I wish to thank Dr. Lucas Dixon and Dr. Ross Duncan for the work they put into Quantomatic, both the theory and the implementation, without which this work could have taken much longer. Additionally, I owe thanks to Matvey Soloviev for his bright ideas in the DCM paper, which really opened up the possibility of reasoning about !-graphs using graph rewriting.

Thanks also to the administrative staff of the Department of Computer Science for helping me navigate university bureaucracy, and especially to Julie Sheppard and Janet Sadler.

Finally, a huge thank you to my family and friends, who have supported and encouraged me over the last four years, through the hard times and the good. I couldn't have done it without you.

Abstract

The aim of this thesis is to present an extension to the string graphs of Dixon, Duncan and Kissinger that allows the finite representation of certain infinite families of graphs and graph rewrite rules, and to demonstrate that a logic can be built on this to allow the formalisation of inductive proofs in the string diagrams of compact closed and traced symmetric monoidal categories.

String diagrams provide an intuitive method for reasoning about monoidal categories. However, this does not negate the ability for those using them to make mistakes in proofs. To this end, there is a project (Quantomatic) to build a proof assistant for string diagrams, at least for those based on categories with a notion of trace. The development of string *graphs* has provided a combinatorial formalisation of string diagrams, laying the foundations for this project.

The prevalence of commutative Frobenius algebras (CFAs) in quantum information theory, a major application area of these diagrams, has led to the use of variable-arity nodes as a shorthand for normalised networks of Frobenius algebra morphisms, so-called "spider notation". This notation greatly eases reasoning with CFAs, but string graphs are inadequate to properly encode this reasoning.

This dissertation firstly extends string graphs to allow for variable-arity nodes to be represented at all, and then introduces !-box notation – and structures to encode it – to represent string graph equations containing repeated subgraphs, where the number of repetitions is abitrary. This can be used to represent, for example, the "spider law" of CFAs, allowing two spiders to be merged, as well as the much more complex generalised bialgebra law that can arise from two interacting CFAs.

This work then demonstrates how we can reason directly about !-graphs, viewed as (typically infinite) families of string graphs. Of particular note is the presentation of a form of graph-based induction, allowing the formal encoding of proofs that previously could only be represented as a mix of string diagrams and explanatory text.

Contents

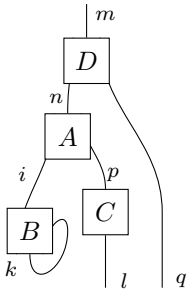
Chapter 1

Introduction

In the early 1950s, as a research student attending lectures on differential geometry, Roger Penrose developed a pictoral notation for dealing with tensors. He started using this in publications some twenty years later, most notably in [32]. His motivation for using this notation was as an exposition and working aid when dealing with the tensor index notation common in various areas of physics. [32], in which Penrose generalised and formalised tensor index notation as *abtract tensors*, is liberally illustrated with these diagrams. In a tensor expression like

$$A^{ip}_{n} B^{k}_{ik} C^{l}_{p} D^{nq}_{m}$$

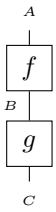
where we can consider the large letters to be morphisms with subscripts as inputs and superscripts as outputs, we have to keep careful track of which indices are repeated (and therefore should be *contracted* – the notion of composition for abstract tensors). This is made clear in the following diagrammatic representation



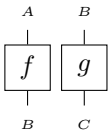
These *string diagrams* (or *wire diagrams*) are particularly good at capturing information with a planar (or mostly planar), rather than linear, structure. Thus it was quite natural that they came to be used for monoidal categories, which have an associative, unital bifunctor $\bigotimes : \mathcal{C} \times \mathcal{C} \to \mathcal{C}$ that acts as a form of composition orthogonal to the categorial composition.

Note that, unlike in the traditional diagrams of category theory, string diagrams represent morphisms as nodes and objects as wires. This allows categorical composition to be represented by

joining wires, as in



while the monoidal composition can be represented via juxtaposition



Twenty years after Penrose's publication, Joyal and Street formalised the link between string diagrams and monoidal categories, at least in part[20]. In particular, they showed that *progressive plane diagrams*, a generalised topological graph with directed wires that do not cross or have loops, exactly capture the axioms of monoidal categories. Additionally, they showed that allowing crossings captures the axioms of symmetric monoidal categories, which have a natural isomorphism witnessing that $A \otimes B \cong B \otimes A$ for all objects A and B, and that other variations on these diagrams capture braided and balanced monoidal categories.

Joyal and Street planned a follow-up paper to treat non-progressive diagrams, such as those with loops, but this was never published. The link between these diagrams and either *traced symmetric monoidal categories* or *compact closed categories* (depending on the restrictions placed on the diagrams) was shown in [23].

We concentrate on traced symmetric monoidal categories and compact closed categories in this thesis. Selinger's survey paper[37] gives a more comprehensive overview of graphical languages for various types of monoidal categories.

These diagrammatic languages found a home in the study of quantum computer science, when in 2004 Samson Abramsky and Bob Coecke recast von Neumann's axiomatisation of quantum mechanics into category theory, allowing the use of string diagrams. The diagrams proved to be particularly helpful in illuminating the role of entanglement in various quantum mechanical and quantum informational protocols, and provided a grounding for intuition in something familiar (the physical manipulation of a diagram) for something that, at many times, appears to behave counter-intuitively (quantum mechanics).

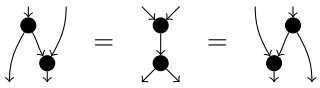
In this setting, the wires of a string diagram represent quantum mechanical systems, and the nodes are operations on those systems. Of particular relevance to this thesis is the importance of commutative Frobenius algebras in these languages.

A Frobenius algebra in a monoidal category $\mathcal{C}$ consists of a monoid (A, μ, η) and a comonoid (A, δ, ϵ) on an object of $\mathcal{C}$ – the monoid being an associative multiplication operation $\mu : A \otimes A \to A$

with unit $\eta : I \to A$, and the comonoid being its dual $\delta : A \to A \otimes A$ with counit $\epsilon : A \to I$ – that interact "nicely". Specifically, they obey the Frobenius law:

$$(1_A \otimes \mu) \circ (\delta \otimes 1_A) = \delta \circ \mu = (\mu \otimes 1_A) \circ (1_A \otimes \delta)$$

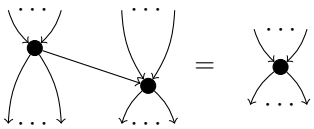
If we represent the monoid and comonoid graphically as $(A,$  $)$ and $(A,$  $)$, then this law can be drawn



Commutative Frobenius algebras (CFAs) are those whose monoid is commutative and comonoid is cocommutative. They have the property that the value of any connected (in the graphical language) network of components of a given CFA is determined purely by the number of inputs, outputs and loops it has. This gives rise to a consise and useful representation of a CFA, the "spider":

Loops are encoded in this representation as self-loops (by connecting outputs to inputs).

This leads to the *spider law*, which allows us to merge two connected spiders of the same type



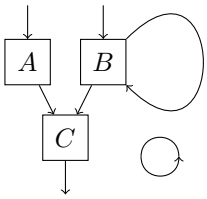
and this law significantly simplifies working with these graphical languages.

While the use of string diagrams can help greatly with understanding and intuition, constructing large proofs by hand can still be tedious and error-prone. This is where projects like Quantomatic[25], a collection of automated tools for graphical languages, come in.

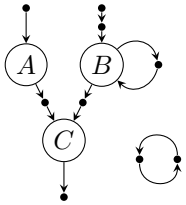
The goal of Quantomatic is to produce a useful proof assistant for graphical languages; as its name suggests, it was originally aimed at the languages being developed for quantum computation. For example, Dixon and Duncan used it to verify the correctness of the Steane code in [12]. It is still primarily used for quantum languages, but it aims to be useful in other fields as well.

Dixon, Duncan and Kissinger provided a theoretical underpinning for Quantomatic's automated reasoning in the form of *string graphs*, introduced as *open graphs* in [10] and further refined by Dixon and Kissinger in [11] and then by Kissinger in [23]. These represent string diagrams as typed graphs, where the vertex types are sorted into *node-vertices* and *wire-vertices*, the former being the "real" vertices from the string diagram and the latter a kind of "dummy" vertex holding the wires

in place (as well as carrying the wire type). For example, the string diagram



could be represented by the graph

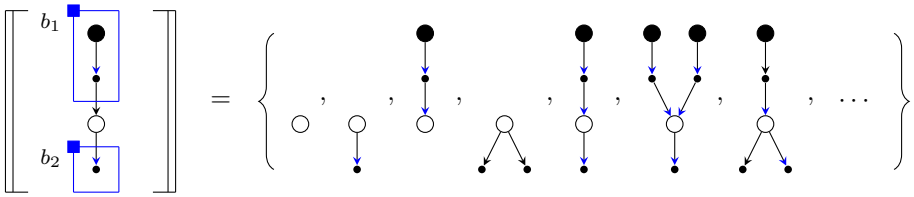


This allows us to do equational reasoning for diagrams using graph rewriting, in a similar manner to how traditional proof assistants use term rewriting.

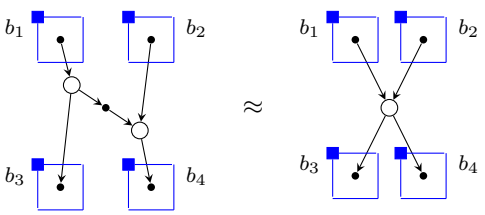
Note that string graphs are not the only combinatorial representation for the categorical structures that underlie string diagrams. In particular, Hasegawa, Hofmann and Plotkin use a bijective map to describe the connections formed by wires in their paper showing that finite dimensional vector spaces are complete for traced symmetric monoidal categories[18]. Using graphs, however, allows us to build on a substantial body of established work on graph rewriting.

The first aim of this dissertation is to extend this formalism to allow tools like Quantomatic to work with the spider law and other, more complex, rules using the !-*graphs* posited in [9] as graph patterns. In particular, we aim to give a mathematical underpinning to these, which were previously only treated informally.

!-graphs are string graphs with demarked subgraphs called !-*boxes* that can be repeated any number of times, allowing an infinite family of graphs to be represented as a single graph:

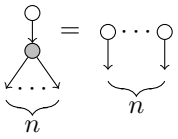


This can be extended to string graph equations, allowing the spider law to be represented as a !-graph equation:

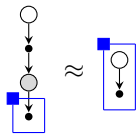


We call a string graph or string graph equation represented by a !-graph or !-graph equation an *instance* of that graph or rule.

This notation may seem overpowered for simply representing the spider law, but it also allows many equations describing interactions between certain kinds of CFA, such as



which can be represented



and even more complex examples that we will describe in later chapters.

The second aim of this dissertation is to provide the beginnings of a formal system for reasoning *about* (and not just *with*) these !-graphs. We show that rewrite rules composed of !-graphs can be used to rewrite !-graphs and, in doing so, perform equational reasoning on these infinite families of string graphs. What is more, we introduce some additional rules to this logic of !-graphs, including a form of graphical induction that encodes inductive proofs that could previously only be done with a mix of diagrams and explanatory text.

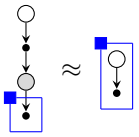
Chapter 2 describes various types of monoidal category and the graphical languages they induce. It also provides some motivating examples in the form of a subset of the Z/X calculus for quantum information processing.

Chapter 3 provides a brief introduction to rewriting as a way of doing equational reasoning, using the well-established field of term rewriting. It then presents an equational logic for traced symmetric monoidal categories and compact closed categories using string graphs, and shows how this can be implemented using string graph rewriting. We present a slightly different construction of string graphs to [23], as Kissinger's construction does not allow for variable-arity nodes, required by the spiders of the quantum graphical languages.

Chapter 4 describes the !-box notation and its encoding in !-graphs. It provides a set of operations on !-boxes (COPY, DROP and KILL) that are used to define the concrete instances of a !-graph (ie: those string graphs it represents).

Chapter 5 introduces !-graph equations to represent families of string graph equations, and demonstrates how their directed form, !-graph rewrite rules, can be used to rewrite not only string graphs but also !-graphs. The latter produces further !-graph equations that are sound with respect to their interpretation as families of string graph equations. This effectively allows us to rewrite a potentially infinite family of string graphs simultaneously.

Chapter 6 builds on this to construct a nascent logic for reasoning about !-graphs. In particular, we construct an analogue of induction for string graphs which functions as a !-box introduction rule, and demonstrate how this can be used to derive



in the Z/X calculus. We also demonstrate a derivation of the generalised bialgebra law, which we introduce at the end of chapter 2.

Chapter 7 demonstrates an algorithm for finding instances of !-graphs that match string graphs or other !-graphs, allowing for practical implementations of rewriting with !-graph rewrite rules. The complete matching process implemented in Quantomatic for rewriting string graphs with !-graph rewrite rules is described, as well as the planned extension to it that will allow !-graphs to be rewritten.

Finally, chapter 8 presents our conclusions and areas of further work.

Chapter 2

Diagrammatic Reasoning

This chapter provides a brief summary from the existing literature of various types of monoidal categories that are relevant to this thesis, and the diagrammatic representations of their morphisms. It also demonstrates their utility in reasoning about such categories using examples drawn from existing work in the area of graphical languages for quantum computation.

2.1 Monoidal Categories

Monoidal categories provide a useful setting for reasoning about processes of all kinds. In addition to the usual functional composition, they allow a form of parallel composition via the tensor product.

Definition 2.1.1 (Monoidal Category)**.** A *monoidal category* is a category $\mathcal{M}$ equipped with

- a bifunctor $\bigotimes : \mathcal{M} \times \mathcal{M} \to \mathcal{M}$, called the *tensor product*;
- a distinguished object I of $\mathcal{M}$, called the *tensor unit*; and
- three natural isomorphisms

$$\alpha_{A,B,C} : (A \otimes B) \otimes C \cong A \otimes (B \otimes C)$$
$$\lambda_A : I \otimes A \cong A$$
$$\rho_A : A \otimes I \cong A$$

where $\lambda_I = \rho_I$ and the following diagrams commute:

$$\begin{array}{ccccc} ((A \otimes B) \otimes C) \otimes D & \xrightarrow{\alpha} & (A \otimes B) \otimes (C \otimes D) & \xrightarrow{\alpha} & A \otimes (B \otimes (C \otimes D)) \\ \downarrow{\scriptstyle \alpha \otimes 1} & & & & \uparrow{\scriptstyle 1 \otimes \alpha} \\ (A \otimes (B \otimes C)) \otimes D & & \xrightarrow{\alpha} & & A \otimes ((B \otimes C) \otimes D) \end{array}$$

$$\begin{array}{ccc} (A \otimes I) \otimes B & \xrightarrow{\alpha} & A \otimes (I \otimes B) \\ & {\scriptstyle \rho \otimes 1} \searrow \quad \swarrow {\scriptstyle 1 \otimes \lambda} & \\ & A \otimes B & \end{array}$$

Example 2.1.2. The category **Set** of sets is monoidal, with the cartesian product as $\bigotimes$ and the single-element set as I. Alternatively, the disjoint union of sets and the empty set can be used as the monoidal structure.

The category $\mathbf{Vect_K}$ of vector spaces over a field K and linear functions between them is a monoidal category, where $\bigotimes$ is the tensor product of vector spaces, and I is K as a one-dimensional vector space.

In later examples, we will use the fact that if U, V and W are vector spaces over K, then each bilinear map $f : U \times V \to W$ induces a unique linear map $\tilde{f} : U \otimes V \to W$. We can use this to construct the natural isomorphisms $\alpha_{U,V,W}$, λ_V and ρ_V from the maps

$$((u, v), w) \mapsto u \otimes (v \otimes w) \qquad (k, v) \mapsto kv \qquad (v, k) \mapsto kv$$

When considering (categorical) diagrams of a monoidal category $\mathcal{M}$ involving the above isomorphisms, Mac Lane's coherence theorem[29] allows us to work instead in a category where those isomorphisms are identities; we call this category the *strictification* of $\mathcal{M}$. Formal categorical diagrams (of the sort seen in definition 2.1.1) commute in $\mathcal{M}$ if and only if they commute in its strictification. Therefore, we abuse notation slightly by omitting paretheses and, insofar as possible, I, writing

$$A \otimes B \otimes C$$

instead of

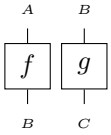
$$(A \otimes I) \otimes (B \otimes C)$$

for example.

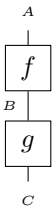
In [32], Penrose presented a graphical language he had developed for representing tensor networks. Restricted variants of this language can be used for talking about more general monoidal categories.

In this notation, we represent objects of the category (and identity morphisms on those objects) as lines, and morphisms as boxes, also called nodes. For example, the morphism $f : A \to B$ could be depicted

where we read the diagram in a downwards direction. The tensor product is depicted by placing things side-by-side, so if $g : B \to C$ then $f \otimes g : A \otimes B \to B \otimes C$ is



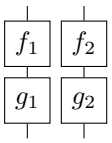
and $g \circ f : A \to C$ is



One convenience that this immediately provides, other than requiring less mental effort to parse than conventional symbolic notation, is that the equation

$$(g_1 \otimes g_2) \circ (f_1 \otimes f_2) = (g_1 \circ f_1) \otimes (g_2 \circ f_2)$$

– a consequence of the fact that $\bigotimes$ is a bifunctor – is implicit in the graphical notation; both sides of the equation are, in fact, represented by the same diagram



The tensor product has a rigid order; it is common to allow more flexibility, in a manner akin to rerouting information. This is provided by a symmetric monoidal category.

Definition 2.1.3 (Symmetric Monoidal Category)**.** A *symmetric monoidal category*, or SMC, is a monoidal category $\mathcal{M}$ equipped with a natural isomorphism

$$\gamma_{A,B} : A \otimes B \cong B \otimes A$$

such that $\gamma_{A,B} \circ \gamma_{B,A} = 1$, $\rho_A = \lambda_A \circ \gamma_{A,I}$ and the following diagram commutes:

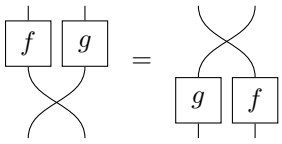
$$\begin{array}{ccccc} (A \otimes B) \otimes C & \xrightarrow{\alpha} & A \otimes (B \otimes C) & \xrightarrow{\gamma} & (B \otimes C) \otimes A \\ \downarrow{\scriptstyle \gamma \otimes 1} & & & & \downarrow{\scriptstyle \alpha} \\ (B \otimes A) \otimes C & \xrightarrow{\alpha} & B \otimes (A \otimes C) & \xrightarrow{1 \otimes \gamma} & B \otimes (C \otimes A) \end{array}$$

Example 2.1.4. Set with either cartesian products or disjoint unions is symmetric, as is $\mathbf{Vect}_{\mathbf{K}}$. In the latter case, γ can be defined (see example 2.1.2) using the map

$$(v, w) \mapsto (w \otimes v)$$

Diagrammatically, we represent $\gamma_{A,B}$ as

The fact that γ is a natural isomorphism is represented in this notation as an ability to "slide" morphisms up and down wires:



Joyal and Street presented[20] a formalisation of Penrose's diagrams based on topological graphs. They showed that *progressive polarised diagrams*, essentially those without loops, can be used to form free symmetric monoidal categories; effectively, they exactly capture the axioms of a symmetric monoidal category.

There are plenty of scenarios where we do want to introduce loops, whether to represent the trace (or partial trace) operation on a matrix, looping in a computation or even the effects of entanglement in a quantum mechanical system, where information can sometimes appear to flow backwards in time.

A *trace operation*, if one exists, can be used to capture a looping construction.

Definition 2.1.5 (Traced Symmetric Monoidal Category)**.** A *traced symmetric monoidal category* is a symmetric monoidal category $\mathcal{M}$ together with a *trace operation*: a function

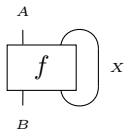
$$\mathrm{Tr}^X_{A,B} : \hom_{\mathcal{M}}(A \otimes X, B \otimes X) \to \hom_{\mathcal{M}}(A, B)$$

for all objects X, A, B that satisfies the following conditions:

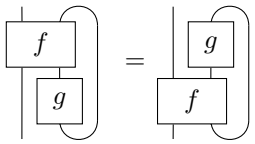
1. $\mathrm{Tr}^X((g \otimes 1_X) \circ f \circ (h \otimes 1_X)) = g \circ \mathrm{Tr}^X(f) \circ h$
2. $\mathrm{Tr}^Y(f \circ (1_A \otimes g)) = \mathrm{Tr}^X((1_B \otimes g) \circ f)$
3. $\mathrm{Tr}^I(f) = f$
4. $\mathrm{Tr}^{X \otimes Y}(f) = \mathrm{Tr}^X(\mathrm{Tr}^Y(f))$
5. $\mathrm{Tr}^X(g \otimes f) = g \otimes \mathrm{Tr}^X(f)$
6. $\mathrm{Tr}^X(\gamma_{X,X}) = 1_X$

Example 2.1.6. $\mathbf{Vect_K}$ is a traced symmetric monoidal category, with the partial trace of linear maps as the trace operation.

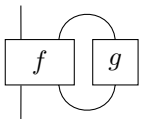
$\mathrm{Tr}^X_{A,B}(f)$ is represented as



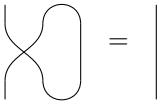
and, as before, the properties of the trace operation are subsumed into its representation in the graphical language, either directly (as with 4 or 5, for example, where both sides of the equation have the same representation) or via "natural" graphical manipulations. For example, condition 2 corresponds to sliding a morphism round a trace loop from one end to the other:



although care must be taken, as the following diagram has no meaning in a traced SMC:



(but see remark 2.1.10 below). Condition 6 is visually represented as an ability to "yank" loops with no (non-identity) morphisms straight:



Finally, we come to *compact closed categories*. These are based around *dual objects*, which are the categorisation of dual spaces from linear algebra.

Definition 2.1.7 (Dual Objects)**.** Let A and A^* be objects in a monoidal category. A^* is called a *left dual* of A (and A a *right dual* of A^*) if there exist morphisms $d_A : A \otimes A^* \to I$ and $e_A : I \to A^* \otimes A$ such that

$$(d_A \otimes 1_A) \circ (1_A \otimes e_A) = 1_A$$

and

$$(1_{A^*} \otimes d_A) \circ (e_A \otimes 1_{A^*}) = 1_{A^*}$$

Note that in a symmetric monoidal category, if A^* is a left dual of A, it is also a right dual of A and vice versa, so we simply say that A^* is the dual of A.

Definition 2.1.8 (Compact Closed Category)**.** A *compact closed category* is a symmetric monoidal category where every object has a dual.

We add the notion of dual objects to the graphical language using arrows. If an object A is represented by a downward-directed wire, its dual is represented by an upward-directed one. The maps d_A and e_A can then be drawn

respectively. The equations in the above definition can then be drawn

Example 2.1.9. Consider a finite-dimensional vector space vector space V and its (algebraic) dual space, V^*. Given a basis $\{u_1, \ldots, u_n\}$ of V, there is a corresponding basis $\{u_1^*, \ldots, u_n^*\}$ of V^* such that

$$u_i^*(a_1 u_1 + \cdots + a_n u_n) = a_i$$

We can then define the map $d_V : V \otimes V^* \to K$ to be induced (see example 2.1.2) by

$$(v, f) \mapsto f(v)$$

and $e_V : K \to V^* \otimes V$ to be

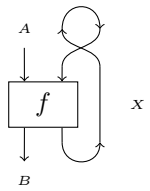
$$k \mapsto k(u_1^* \otimes u_1 + \cdots + u_n^* \otimes u_n)$$

So if we consider V as an object of $\mathbf{FDVect_K}$, the category of finite-dimensional vector spaces over K, V^* is a categorical dual of V. Thus $\mathbf{FDVect_K}$ is compact closed (although $\mathbf{Vect_K}$ is not). Note, however, that our definitions of d_V and e_V are not the only possibilities.

Given the graphical notation, we would expect to be able to construct a trace operation using these maps; this is indeed the case[21]. $\mathrm{Tr}^X_{A,B}(f)$ is

$$(1_B \otimes d_X) \circ (f \otimes 1_{X^*}) \circ (1_A \otimes e_X)$$

or, graphically,



Remark 2.1.10. Joyal, Street and Verity's Int construction[21] allows a traced SMC to be embedded in a compact closed category with formal dual objects. This allows the diagrams for compact closed categories to be given meaning when working with traced SMCs.

2.2 Quantum Computation with Diagrams

An area which is making active use of these diagrammatic languages is quantum information processing. Starting with a paper by Abramsky and Coecke in 2004[1], there has been considerable work in axiomatising quantum mechanics in a categorical setting that admits diagrammatic reasoning. Indeed, a major application of the ideas set out in this thesis is to make this work easier. We will therefore present a brief introduction to one of these languages, both as motivation and as a source of examples.

Most of the languages that have been developed in this area are based on compact closed categories; in particular, the usual model for them is the category **FDHilb** of finite dimensional *Hilbert spaces*.

Definition 2.2.1. A *Hilbert space* is a real or complex inner product space that is a complete metric space with respect to the distance metric

$$d(x, y) = \sqrt{\langle x - y, x - y \rangle}$$

Note that we can ignore the compactness requirement, as it is satisfied by all finite-dimensional inner product spaces.

FDHilb is a compact closed category, with the categorical dual of a Hilbert space H being the algebraic dual H^*, as in **FDVect** (example 2.1.9). As such, we can make use of the diagrammatic language already introduced for such categories.

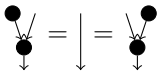
A structure that has particular importance in this formulation of quantum mechanics is the Frobenius algebra, a monoid and comonoid pair that interact particularly well.

A *monoid* in a monoidal category $\mathcal{C}$ as a triple (A, μ, η) where A is an object of $\mathcal{C}$, and $\mu : A \otimes A \to A$ and $\eta : I \to A$ are morphisms satisfying unit and associativity laws:

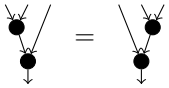
$$\mu \circ (\eta \otimes 1_A) \circ \lambda_A^{-1} = 1_A = \mu \circ (1_A \otimes \eta) \circ \rho_A^{-1}$$

$$\mu \circ (\mu \otimes 1_A) = \mu \circ (1_A \otimes \mu)$$

Graphically, we can represent the monoid as $(A, $  $)$; these laws can then be drawn



and

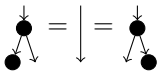


A *comonoid* is the dual of this: a triple (A, δ, ϵ) with $\delta : A \to A \otimes A$ and $\epsilon : A \to I$ satisfying counit and coassociativity laws:

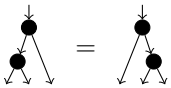
$$\lambda_A \circ (\epsilon \otimes 1_A) \circ \delta = 1_A = \rho_A \circ (1_A \otimes \epsilon) \circ \delta$$

$$(\delta \otimes 1_A) \circ \delta = (1_A \otimes \delta) \circ \delta$$

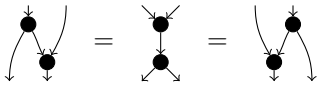
Graphically, this is $(A, $  $)$ satisfying



and



A monoid and comonoid on the same object form a *Frobenius algebra* if they satisfy the Frobenius law:



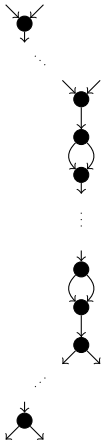
or, symbolically,

$$(1_A \otimes \mu) \circ (\delta \otimes 1_A) = \delta \circ \mu = (\mu \otimes 1_A) \circ (1_A \otimes \delta)$$

A Frobenius algebra is *commutative* if the monoid is commutative and the comonoid is cocommutative:

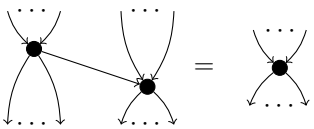
$$\mu = \mu \circ \gamma_{A,A} \qquad\qquad \delta = \gamma_{A,A} \circ \delta$$

Commutative Frobenius algebras (CFAs) appear repeatedly in quantum graphical languages, whether to describe entanglement[5, 6] or classical data[8]. Thus the practical utility of these languages is greatly improved by a consequence of the Frobenius law, namely that any connected (in the diagrammatic language) network of multiplications, comultiplications, units and counits of a commutative Frobenius algebra has a unique normal form consisting of a series of multiplications followed by a series of loops followed by a series of comultplications:



This allows a more compact representation, generally referred to as a "spider":

where the loops are encoded by connecting outputs of the spider to inputs. Note that our original multiplication, comultiplication, unit and counit are subsumed into this notation. This leads to the *spider law*, which allows us to merge two connected spiders of the same type



The spider law and the spider identity law

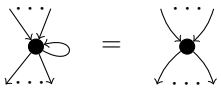
completely capture the laws of a CFA.

There are various quantum graphical languages built around CFAs, including the Z/X calculus[5], the trichromatic calculus[28] and the GHZ/W calculus[6]. There are also several languages built on top of the Z/X calculus, such as Duncan and Perdrix's calculus for measurement-based quantum computation[13]. The GHZ/W calculus even admits an encoding of rational arithmetic[7].

We will use a restricted version of the Z/X calculus (without angles or scalars) to provide motivating examples. This consists of two *special* CFAs, called the X and Z CFAs, together with certain rules governing their interaction.

Definition 2.2.2. A CFA ($\mu, \eta, \delta, \epsilon$) is *special* if it satisfies

The "spiderised" version of this is



Traditionally, the X CFA has been represented with red nodes, and the Z CFA with green nodes. However, we will follow [23] in using grey for X and white for Z, if only to spare colourblind readers some frustration. Thus our CFAs are

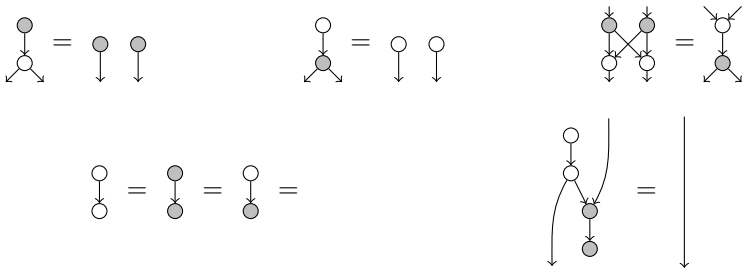
$$(\mu_X, \eta_X, \delta_X, \epsilon_X) = (\mu, \eta, \delta, \epsilon)$$

and

$$(\mu_Z, \eta_Z, \delta_Z, \epsilon_Z) = (\mu, \eta, \delta, \epsilon)$$

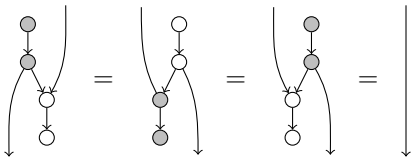
.

In addition to the laws of a special CFA for each colour of node, we have the following laws:

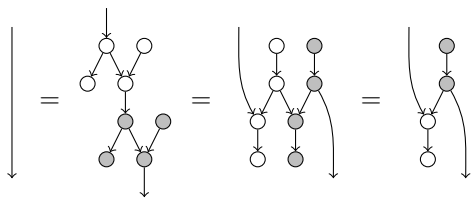


We name these laws, for ease of reference, *Z copies X*, *X copies Z*, the *bialgebra law*, *scalar elimination* and the *dual law*, in order.

For the dual law we might expect certain symmetries:



Indeed, we can use the laws of a CFA together with the dual law to derive these. For example, the last one can be derived as follows:



This fairly simple proof can be written in "traditional" mathematical notation as well, but – because we have to explicitly use various laws and axioms that are inherent in the diagrammatic language – it ends up being long and tedious. We present such a proof below to demonstrate this. For reference, the dual law written symbolically is

$$\rho \circ (1_A \otimes (\epsilon_X \circ \mu_X)) \circ ((\delta_Z \circ \eta_Z) \otimes 1_A) \circ \lambda^{-1} = 1_A$$

Then we have

$$\begin{aligned}
1_A =& \{\text{(co)unitality laws}\} \\
& \mu_X \circ (1_A \otimes \eta_X) \circ \rho^{-1} \circ \lambda \circ (\epsilon_X \otimes 1_A) \circ \delta_X \circ \\
& \mu_Z \circ (1_A \otimes \eta_Z) \circ \rho^{-1} \circ \lambda \circ (\epsilon_Z \otimes 1_A) \circ \delta_Z \\
=& \{\text{naturality of } \lambda, \rho\} \\
& \lambda \circ (1_I \otimes \mu_X) \circ (1_I \otimes 1_A \otimes \eta_X) \circ (\epsilon_X \otimes 1_A \otimes 1_I) \circ (\delta_X \otimes 1_I) \circ \rho^{-1} \circ \\
& \lambda \circ (1_I \otimes \mu_Z) \circ (1_I \otimes 1_A \otimes \eta_Z) \circ (\epsilon_Z \otimes 1_A \otimes 1_I) \circ (\delta_Z \otimes 1_I) \circ \rho^{-1} \\
=& \{\text{bifunctoriality of } \otimes\} \\
& \lambda \circ (\epsilon_X \otimes 1_A) \circ (1_A \otimes \mu_X) \circ (\delta_X \otimes 1_A) \circ (1_A \otimes \eta_X) \circ \rho^{-1} \circ \\
& \lambda \circ (\epsilon_Z \otimes 1_A) \circ (1_A \otimes \mu_Z) \circ (\delta_Z \otimes 1_A) \circ (1_A \otimes \eta_Z) \circ \rho^{-1} \\
=& \{\text{Frobenius law}\} \\
& \lambda \circ (\epsilon_X \otimes 1_A) \circ (\mu_X \otimes 1_A) \circ (1_A \otimes \delta_X) \circ (1_A \otimes \eta_X) \circ \rho^{-1} \circ \\
& \lambda \circ (\epsilon_Z \otimes 1_A) \circ (\mu_Z \otimes 1_A) \circ (1_A \otimes \delta_Z) \circ (1_A \otimes \eta_Z) \circ \rho^{-1} \\
=& \{\text{bifunctoriality of } \otimes\} \\
& \lambda \circ ((\epsilon_X \circ \mu_X) \otimes 1_A) \circ (1_A \otimes (\delta_X \circ \eta_X)) \circ \rho^{-1} \circ \\
& \lambda \circ ((\epsilon_Z \circ \mu_Z) \otimes 1_A) \circ (1_A \otimes (\delta_Z \circ \eta_Z)) \circ \rho^{-1} \\
=& \{\text{naturality of } \lambda, \rho\} \\
& \lambda \circ (1_I \otimes \lambda) \circ (1_I \otimes (\epsilon_X \circ \mu_X) \otimes 1_A) \circ (1_I \otimes 1_A \otimes (\delta_X \circ \eta_X)) \circ \\
& ((\epsilon_Z \circ \mu_Z) \otimes 1_A \otimes 1_I) \circ (1_A \otimes (\delta_Z \circ \eta_Z) \otimes 1_I) \circ (\rho^{-1} \otimes 1_I) \circ \rho^{-1} \\
=& \{\text{bifunctoriality of } \otimes\} \\
& \lambda \circ ((\epsilon_Z \circ \mu_Z) \otimes 1_A) \circ (1_A \otimes 1_A \otimes \lambda) \circ (1_A \otimes 1_A \otimes (\epsilon_X \circ \mu_X) \otimes 1_A) \circ \\
& (1_A \otimes (\delta_Z \circ \eta_Z) \otimes 1_A \otimes 1_A) \circ (\rho^{-1} \otimes 1_A \otimes 1_A) \circ (1_A \otimes (\delta_X \circ \eta_X)) \circ \rho^{-1} \\
=& \{\text{laws of } \lambda, \rho\} \\
& \lambda \circ ((\epsilon_Z \circ \mu_Z) \otimes 1_A) \circ (1_A \otimes \rho \otimes 1_A) \circ (1_A \otimes 1_A \otimes (\epsilon_X \circ \mu_X) \otimes 1_A) \circ \\
& (1_A \otimes (\delta_Z \circ \eta_Z) \otimes 1_A \otimes 1_A) \circ (1_A \otimes \lambda^{-1} \otimes 1_A) \circ (1_A \otimes (\delta_X \circ \eta_X)) \circ \rho^{-1}
\end{aligned}$$

$=$\{bifunctoriality of $\otimes$\}

$\lambda \circ ((\epsilon_Z \circ \mu_Z) \otimes 1_A) \circ$

$(1_A \otimes (\rho \circ (1_A \otimes (\epsilon_X \circ \mu_X)) \circ ((\delta_Z \circ \eta_Z) \otimes 1_A) \circ \lambda^{-1}) \otimes 1_A) \circ$

$(1_A \otimes (\delta_X \circ \eta_X)) \circ \rho^{-1}$

$=$\{dual law\}

$\lambda \circ ((\epsilon_Z \circ \mu_Z) \otimes \lambda) \circ$

$(1_A \otimes 1_A \otimes 1_A) \circ$

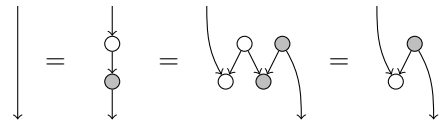
$(\rho^{-1} \otimes (\delta_X \circ \eta_X)) \circ \rho^{-1}$

$=$\{identity\}

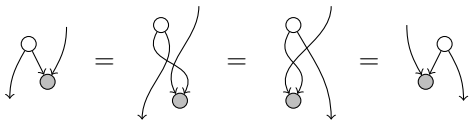
$\lambda \circ ((\epsilon_Z \circ \mu_Z) \otimes \lambda) \circ (\rho^{-1} \otimes (\delta_X \circ \eta_X)) \circ \rho^{-1}$

Even with explanatory notes about what law is being used when, this proof is hard to follow – and was hard to construct – compared to the graphical notation.

Note that we can make the graphical version even cleaner by using spider notation:



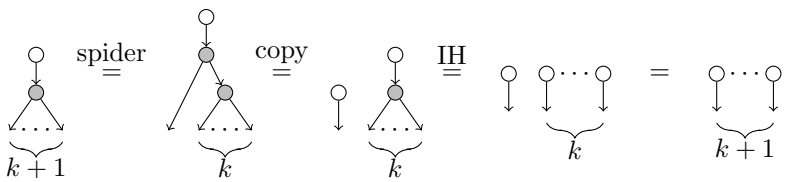
The other two symmetries rely on commutativity. In particular, we have



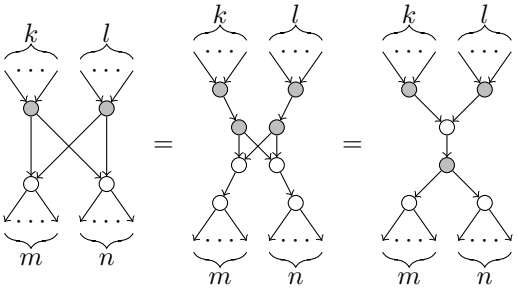
and similarly with the colours swapped.

We can create "spiderised" versions of the copying laws

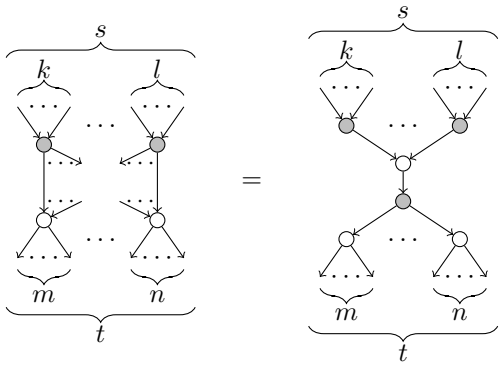
and we can prove that these are correct by induction on n. The proof is fairly simple, and we will just do the X copies Z version, since the other is symmetric. The base case ($n = 0$) follows by the scalar elimination laws. So consider $n = k + 1$:



We can construct a similarly spiderised version of the bialgebra law with some trivial applications of the spider law



but there is also a generalised version of this involving the (s, t)-bipartite graph of X and Z spiders (the normal bialgebra law is the case when $s = t = 2$). Graphically, this looks something like



but it is hard to see what this is supposed to represent without an accompanying textual description. For the same reason, proving this graphically is a messy affair, and convincing yourself that such a proof is correct is even harder.

Chapters 4 and 5 will introduce a formal notation capable of expressing this generalised bialgebra law, and chapter 6 will provide the techniques necessary to prove it formally, together with an informal demonstration of the proof.

Chapter 3

Rewriting

Equational reasoning is core to many problems in mathematics and computer science, from proving that a left unit is a right unit in a commutative ring to optimising functional programs.

The traditional approach to mechanising equational reasoning is to use term rewriting to implement equational logic. We will briefly describe this approach before presenting a related mechanisation scheme, adapted from the existing literature (and [23] in particular), that is much more suited to the examples in the preceeding chapter.

3.1 Term Rewriting and the Word Problem

In order to reason formally about mathematical equations, we need a symbolic language to express those equations. This section summarises the relevant parts of a large body of existing literature on term rewriting and universal algebra; a more comprehensive introduction can be found in [3].

A *signature* provides symbols for the "fixed" elements of a theory: for example, a multiplication operator or its unit. It is simply a set of symbols Σ, each with an associated non-negative integer known as the arity[1]. In the case of group theory, we might have $\Sigma = \{e, i, m\}$, with respective arities 0, 1 and 2.

If we also have a set of *variables* V disjoint from Σ, we can start to write down *terms* such as $m(i(x), e)$ (where x is a variable). The set of possible Σ-terms over V, $T(\Sigma, V)$, is defined inductively:

- if $x \in V$ then $x \in T(\Sigma, V)$
- if $f \in \Sigma$ with arity n and $t_1, \ldots, t_n \in T(\Sigma, V)$ then $f(t_1, \ldots, t_n) \in T(\Sigma, V)$

We can now encode equations using terms. A Σ-*equation* is just a pair of terms that we wish to consider equal. We write such equations $s \approx t$, where s and t are Σ-terms, to make it clear that s and t are not the same term, but should be considered equal in the theory.

[1] For simplicity, we are presenting *untyped* terms here

The axioms of our theory, for example the axioms of a group, are encoded as a set of Σ-equations, and equational logic[17] describes how we can use those axioms to derive further equalities via the following inference rules, adapted for the notion of signature we are using in [3]

$$(\textsc{Axiom})\ \frac{s \approx t \in E}{E \vdash s \approx t} \qquad (\textsc{Refl})\ \frac{}{E \vdash t \approx t} \qquad (\textsc{Sym})\ \frac{E \vdash s \approx t}{E \vdash t \approx s}$$

$$(\textsc{Trans})\ \frac{E \vdash s \approx t \qquad E \vdash t \approx u}{E \vdash s \approx u} \qquad (\textsc{Subst})\ \frac{E \vdash s \approx t}{E \vdash \sigma(s) \approx \sigma(t)}$$

$$(\textsc{Leibniz})\ \frac{E \vdash s_1 \approx t_1 \qquad \ldots \qquad E \vdash s_n \approx t_n}{E \vdash f(s_1, \ldots, s_n) \approx f(t_1, \ldots, t_n)}$$

where σ is a substitution that replaces variables with other terms and f is an n-ary function symbol of Σ.

We commonly wish to determine whether two Σ-terms s and t are equal by this logic under a certain set of axioms E; in other words, is it the case that $E \vdash s \approx t$? This is known as the *word problem* and is a major application of term rewriting.

The obvious way to approach the word problem is to attempt to use the rules to construct a proof that two terms are equal directly. Term rewriting, in constrast, turns one term into another term that is equal according to the Axiom, Subst and Leibniz rules. This is done by directing equations $s \approx t$ to make them *rewrite rules*, written $s \rightarrowtriangle t$.

A set of terms E can be turned into a *term rewrite system* R in this way, and we define the relation $\rightarrowtriangle_R$ such that $s \rightarrowtriangle_R t$ if and only if there is a substitution σ and a rewrite rule $l \rightarrowtriangle r$ in R such that $\sigma(l)$ occurs in s, and replacing that occurrence by $\sigma(r)$ yields t. For example, if $m(e, x) \rightarrowtriangle x \in R$, then we can deduce that

$$i(m(e, i(y))) \rightarrowtriangle_R i(i(y))$$

Matching is the process of finding an appropriate substitution σ and locating occurences of $\sigma(l)$ in s, and is decidable. Hence, given a finite set of rewrite rules R and a term s, it is possible to find all terms t such that $s \rightarrowtriangle_R t$.

The utility of this approach comes from the observation that the reflexive, symmetric, transitive closure of $\rightarrowtriangle_R$ – which we write $\overset{*}{\leftrightarrowtriangle}_R$ – corresponds exactly to the $\approx$ of equational logic; in other words, if we allow ourselves to view R as a set of equations,

$$s \overset{*}{\leftrightarrowtriangle}_R t \quad \Leftrightarrow \quad R \vdash s \approx t$$

This observation means, in particular, that if we can find a common term u such that $s \overset{*}{\rightarrowtriangle}_R u$ and $t \overset{*}{\rightarrowtriangle}_R u$ (where $\overset{*}{\rightarrowtriangle}_R$ is the transitive closure of $\rightarrowtriangle_R$), then we can deduce that $R \vdash s \approx t$. For term rewrite systems that are *terminating* (there is no infinite chain $s_1 \rightarrowtriangle_R s_2 \rightarrowtriangle_R \ldots$) and *confluent*

(if $s \overset{*}{\twoheadrightarrow}_R t_1$ and $s \overset{*}{\twoheadrightarrow}_R t_2$ then there is a term u such that $t_1 \overset{*}{\twoheadrightarrow}_R u$ and $t_2 \overset{*}{\twoheadrightarrow}_R u$), the word problem is decidable.

While terms, which can have multiple inputs (the variables) but only one output, are well-suited for encoding algebras, they are less useful for structures like coalgebras, where we can have multiple outputs. For example, while a multiplication map $m : A \otimes A \to A$ in a monoidal category is easy to represent as a binary term, a comultiplication map $d : A \to A \otimes A$ does not have such a clear symbolic representation.

Taking our cue from the diagrams of chapter 2, we will use graphs to construct a combinatorial symbolic representation of the sort of maps found in a monoidal category.

3.2 Monoidal Signatures

As with term rewriting, we need a way to represent items of interest from the theory we are reasoning about; in this case, these items correspond to objects and morphisms in a monoidal category.

Definition 3.2.1 (Monoidal Signature: [23], pp 30)**.** A (small, strict) *monoidal signature* $(O, M, \mathrm{dom}, \mathrm{cod})$ consists of two sets M and O together with a pair of functions $\mathrm{dom}, \mathrm{cod} : M \to O^*$ into lists of elements of O.

The intended interpretation of a monoidal signature is that the elements of O are representations of objects of a monoidal category, M is (some of) its morphisms and dom and cod give the types of the domain and codomain of the morphisms. An empty list corresponds to I, the tensor unit.

Definition 3.2.2 (Monoidal Homomorphism: [23], pp 31)**.** If S and T are monoidal signatures, a *monoidal signature homomorphism* $f : S \to T$ consists of functions $f_O : O_S \to O_T$ and $f_M : M_S \to M_T$ making the following diagrams commute:

$$\begin{array}{ccc} M_S & \xrightarrow{\mathrm{dom}_S} & O_S^* \\ {\scriptstyle f_M}\downarrow & & \downarrow{\scriptstyle f_O^*} \\ M_T & \xrightarrow[\mathrm{dom}_T]{} & O_T^* \end{array} \qquad \begin{array}{ccc} M_S & \xrightarrow{\mathrm{cod}_S} & O_S^* \\ {\scriptstyle f_M}\downarrow & & \downarrow{\scriptstyle f_O^*} \\ M_T & \xrightarrow[\mathrm{cod}_T]{} & O_T^* \end{array}$$

Recall from chapter 2 the "spiders" that arise from Frobenius algebras. These are families of morphisms that differ only in how many inputs or outputs of a certain type they have. Rather than having an infinite monoidal signature with an object in M for each possible combination of inputs and outputs, it would be useful to have a single symbol with variable arity. To this end, we adapt the definition of a monoidal signature:

Definition 3.2.3 (Compressed Monoidal Signature)**.** If O and M are sets and $\mathrm{dom}, \mathrm{cod} : M \to (O \times \{\infty, \bullet\})^*$ are functions into lists of pairs of elements of O and either ∞ or $\bullet$, then $(O, M, \mathrm{dom}, \mathrm{cod})$ is a *compressed monoidal signature.*

∞ and $\bullet$ are intended to indicate whether the input or output is of variable or fixed (single, in fact) arity. If it has variable arity, it can be removed or duplicated.

Of course, if this is to be a true compressed representation of a larger monoidal signature, we need a way to reconstruct that monoidal signature from the compressed version.

Definition 3.2.4 (Expansion)**.** Let $S = (O, M, \mathrm{dom}, \mathrm{cod})$ be a compressed monoidal signature.

For each $f \in M$, let D_f be the indices of the ∞-tagged elements of $\mathrm{dom}(f)$, and C_f the indices of the ∞-tagged elements of $\mathrm{cod}(f)$. The *expansion* of S, $\mathrm{expn}(S)$, is $(O, M', \mathrm{dom}', \mathrm{cod}')$ where

- $M' = \bigcup\{\{f\} \times \mathbb{N}^{D_f} \times \mathbb{N}^{C_f} : f \in M\}$
- $\mathrm{dom}'(f, d_f, c_f)$ be $\mathrm{dom}(f)$ with, for all indices i, the ith element replaced with x if it is $(x, \bullet)$ and $d_f(i)$ copies of x if it is (x, ∞)
- $\mathrm{cod}'(f, d_f, c_f)$ be $\mathrm{cod}(f)$ with, for all indices i, the ith element replaced with x if it is $(x, \bullet)$ and $c_f(i)$ copies of x if it is (x, ∞)

S is said to be a compressed monoidal signature for a monoidal signature T if the expansion of S is isomorphic to T. Compressed monoidal signatures that make no use of ∞ correspond exactly to ordinary monoidal signatures.

Example 3.2.5. Consider the compressed monoidal signature where

$$\begin{aligned} O &= \{A, B\} \\ M &= \{f, g\} \\ \mathrm{dom}(f) &= [(A, \infty)] \\ \mathrm{dom}(g) &= [(A, \bullet), (B, \infty)] \\ \mathrm{cod}(f) &= [(A, \bullet), (A, \bullet)] \\ \mathrm{cod}(g) &= [(A, \bullet)] \end{aligned}$$

To make the description simpler, and more evocative of the intended interpretation, we use the following notation:

$$\begin{aligned} f &: [A^\infty] \to [A^\bullet, A^\bullet] \\ g &: [A^\bullet, B^\infty] \to [A^\bullet] \end{aligned}$$

The expansion of this signature would include

$$\begin{aligned} (f, 1 \mapsto 0, \varnothing) &: [\,] \to [A, A] \\ (f, 1 \mapsto 1, \varnothing) &: [A] \to [A, A] \\ (f, 1 \mapsto 2, \varnothing) &: [A, A] \to [A, A] \\ &\vdots \end{aligned}$$

and

$$(g, \varnothing, 2 \mapsto 0) : [A] \to [A]$$
$$(g, \varnothing, 2 \mapsto 1) : [A, B] \to [A]$$
$$(g, \varnothing, 2 \mapsto 2) : [A, B, B] \to [A]$$
$$\vdots$$

Since the elements of a compressed monoidal signature are meant to represent families of morphisms in a category, we need a way of determining *which* morphisms they are supposed to represent. This is provided by a *valuation*.

Note: if X is a list, we use $\mathcal{I}(X)$ to refer to the set of indices of X.

Definition 3.2.6 (Arity Count)**.** Let $(M, O, \mathrm{dom}, \mathrm{cod})$ be a compressed monoidal signature and $f \in M$. Then an *arity count for* f is a pair of maps $d : \mathcal{I}(\mathrm{dom}(f)) \to \mathbb{N}$ and $c : \mathcal{I}(\mathrm{cod}(f)) \to \mathbb{N}$ such that the index of each $\bullet$-tagged element maps to 1.

Definition 3.2.7 (Valuation)**.** A *valuation* v of a compressed monoidal signature $(M, O, \mathrm{dom}, \mathrm{cod})$ over a monoidal category $\mathcal{C}$ is a map v_o from O to the objects of $\mathcal{C}$ together with a map v_m^f for each $f \in M$ that takes arity counts for f to morphisms of $\mathcal{C}$ with

$$v_m^f(d, c) : v_o(\alpha_1)^{d(1)} \otimes \ldots \otimes v_o(\alpha_l)^{d(l)} \to v_o(\beta_1)^{c(1)} \otimes \ldots \otimes v_o(\beta_n)^{c(n)}$$

where $\mathrm{dom}(f) = [\alpha_1, \ldots, \alpha_l]$, $\mathrm{cod}(f) = [\beta_1, \ldots, \beta_n]$, A^0 is the monoidal identity I, and $A^{n+1} = A \otimes A^n$.

The valuation is said to be *expansion-order invariant* if $\mathcal{V}$ is a symmetric monoidal category and for all $f \in M$ and all arity counts (d, c) for f, if i is the index of a variable-arity entry of $\mathrm{dom}(f)$ and $1 \leq j < d(i)$, then

$$v_m^f(d, c) \circ \mathrm{swap}_{i,j} = v_m^f(d, c)$$

where $\mathrm{swap}_{i,j}$ is the map

$$1_{X_1^{d(1)}} \otimes \ldots \otimes 1_{X_{i-1}^{d(i-1)}} \otimes 1_{X_i^{j-1}} \otimes \gamma_{X_i, X_i} \otimes 1_{X_i^{d(i)-j-1}} \otimes 1_{X_{i+1}^{d(i+1)}} \otimes \ldots \otimes 1_{X_l^{d(l)}}$$

where $X_i = v_o(\alpha_i)$, and

$$\mathrm{swap}'_{i',j'} \circ v_m^f(d, c) = v_m^f(d, c)$$

where i', j' and $\mathrm{swap}'_{i',j'}$ are defined analagously for $\mathrm{cod}(f)$ and c.

Expansion-order invariance is intended to capture the idea that, in the expansion of a compressed monoidal signature, a morphism should be commutative on the inputs derived from the same ∞-tagged input, and cocommutative on outputs derived from the same ∞-tagged output.

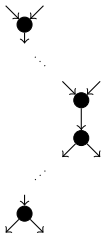
Example 3.2.8. Suppose we have a CFA (page 13) on an object X of a symmetric monoidal category

$$\mu : X \otimes X \to X$$
$$\eta : I \to X$$
$$\delta : X \to X \otimes X$$
$$\epsilon : X \to I$$

and the compressed monoidal signature

$$s : [A^\infty] \to [A^\infty]$$

If we want s to represent the spiderised version of the CFA, we can construct a valuation v such that $v_o(A) = X$ and $v^s_m(1 \mapsto k, 1 \mapsto l)$ is $k-1$ copies of μ followed by $l-1$ copies of δ, arranged in the following form:



If $k = 0$, we use η to get a domain of I, and if $l = 0$ we use ϵ to get a codomain of I.

This valuation is expansion-order invariant because CFAs are (co-)commutative and (co-)associative, so precomposing or postcomposting the map above with swap maps does not affect the value of the morphism.

3.3 Graphs

A well-established categorical construction for directed graphs is to use functors into **Set**, the category of (small) sets:

Definition 3.3.1. Graph is the category of functors from the category

$$E \underset{\text{t}}{\overset{\text{s}}{\rightrightarrows}} V$$

to **Set**.

Each element of **Graph** therefore consists of two sets, one of edges (the image of E) and one of vertices (the image of V), and a pair of functions selecting the source (s) and target (t) of each edge. We say that an edge is *incident* to both its source and its target, and call the set of edges incident to a vertex $v \in V$ the *edge neighbourhood* of v, $E(v)$. Also, the source of an edge is *adjacent* to its target, and vice versa.

We can separate vertices and edges into types through the use of a *typegraph*. This also allows the possible sources and targets of the edges to be constrained. For a given graph $\mathcal{G}$, we can construct the category $\mathbf{Graph}/\mathcal{G}$, the slice category over $\mathcal{G}$. The elements of this category can be viewed as pairs (H, t_H) of a graph H and a graph morphism $t_H : H \to \mathcal{G}$, known as the *typing morphism*, and the morphisms of the category are those graph morphisms $f : G \to H$ that respect the typing morphisms, in the sense that

$$\begin{array}{ccc} G & \xrightarrow{\quad f \quad} & H \\ & {\scriptstyle \tau_G} \searrow \quad \swarrow {\scriptstyle \tau_H} & \\ & \mathcal{G} & \end{array}$$

commutes. We call (H, t_H) a *$\mathcal{G}$-typed graph*.

If $\mathcal{G}$ is a subgraph of $\mathcal{H}$, then any $\mathcal{G}$-typed graph can be viewed as a $\mathcal{H}$-typed graph, simply by reinterpreting the codomain of the typing function (or, equivalently, composing the typing function with the embedding morphism from $\mathcal{G}$ to $\mathcal{H}$). The general theorem, stated in categorical terms, is as follows:

Theorem 3.3.2. *Let C be a category, and $m : S \to T$ a monomorphism in C. Then m induces a full embedding functor $E_m : C/S \to C/T$. If C has pullbacks along monomorphisms, E_m has a right adjoint U_m, and $U_m \circ E_m$ is the identity functor.*

Proof. If (X, τ_X) is an object of C/S, we define $E_m(X, \tau_X) = (X, m \circ \tau_X)$, and let E_m be the identity on morphisms. Recall that a morphism $f : X \to Y$ of C is a morphism $f : (X, \tau_X) \to (Y, \tau_Y)$ of C/S if and only if

$$\tau_Y \circ f = \tau_X$$

But if this is the case, then we must have

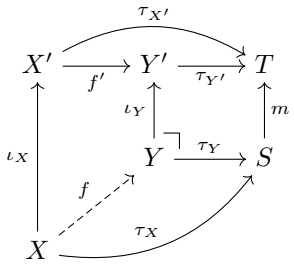
$$(m \circ \tau_Y) \circ f = m \circ \tau_X$$

and hence E_m is a well-defined functor. Since m is monic, the converse is also true, and so E_m is full.

Now suppose C has pullbacks along monomorphisms; we need to show that E_m has a right adjoint $U_m : C/T \to C/S$. Given an object $(X', \tau_{X'})$ in C/T, we define $U_m(X', \tau_{X'})$ to be (X, τ_X) in the following pullback:

$$\begin{array}{ccc} X' & \xrightarrow{\tau_{X'}} & T \\ {\scriptstyle \iota_X}\uparrow & \lrcorner & \uparrow{\scriptstyle m} \\ X & \xrightarrow[\tau_X]{} & S \end{array}$$

which exists since m is a monomorphism. Now let $f' : X' \to Y'$ be a morphism of C/T. We define $U_m(f')$ to be f in the following diagram, which exists and is unique by pullback:



The natural transformations $\eta_{(X,\tau_X)} = 1_X$ and $\epsilon_{(X',\tau_{X'})} = \iota_X$ witness that U_m is right adjoint to E_m, and that $U_m \circ E_m = 1_{C/S}$. □

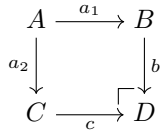
Remark 3.3.3. Note that it is the case that, for any graph T, monomorphisms are exactly injective functions in both **Graph** and **Graph**$/T$. It follows that if $m : S \to T$ is a graph monomorphism, E_m and U_m in the above theorem preserve and reflect monomorphisms.

We will sometimes, as in the next proposition, use the term *pushout of monomorphisms*; this refers to the pushout of a span of monomorphisms, as distinct from a *pushout along a monomorphism* where only one arrow in the span needs to be monic.

Proposition 3.3.4. *Let S and T be graphs and $m : S \to T$ a monomorphism. Then the functors E_m : **Graph**$/S \to$ **Graph**$/T$ and U_m : **Graph**$/T \to$ **Graph**$/S$ from theorem 3.3.2 preserve pushouts of monomorphisms.*

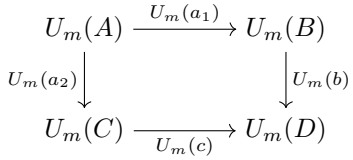
Proof. All left adjoint functors preserve pushouts, and so E_m does in particular.

Consider the following pushout of monomorphisms in **Graph**$/T$:

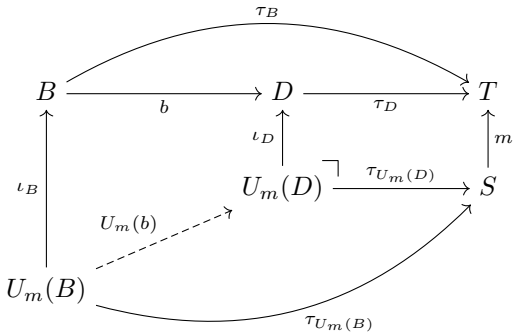


By the **Set**-based construction of **Graph**, we know that the above is a pushout if and only if D is covered by $\mathrm{im}(b)$ and $\mathrm{im}(c)$ and the intersection of those images is exactly the image of $b \circ a_1 = c \circ a_2$ (note that this is only true because all the arrows in the diagram are monomorphisms).

We know that the following diagram commutes and consists of monomorphisms in C/S:



Recall the construction of, for example, the map $U_m(b)$:



It suffices to show that, for each $x \in U_m(D)$, $x \in \text{im}(U_m(b))$ if and only if $\iota_D(x) \in \text{im}(b)$ and similarly for the other morphisms of the pushout. The argument is the same for each morphism, so we will only treat b. We can see from the left square of the diagram that if $x \in \text{im}(U_m(b))$, then $\iota_D(x) \in \text{im}(b)$.

For the converse, suppose $\iota_D(x) \in \text{im}(b)$, and let y be its (unique) preimage under b. Now, $\tau_D(\iota_D(x)) \in \text{im}(m)$, by the pullback square defining $U_m(D)$. So then $\tau_B(y) \in \text{im}(m)$, since $\tau_D \circ b = \tau_B$. So $y \in \text{im}(\iota_B)$ by the pullback defining $U_m(B)$ and its preimage must map to x under $U_m(b)$. Hence $x \in \text{im}(U_m(b))$. □

3.3.1 String Graphs

Arbitrary graphs, however, are too general for our purposes. They have no concept of inputs or outputs, and so there is no clear way to represent a morphism. We therefore make use of *string graphs*, introduced as *open graphs* in [10] and further refined in [23]. These separate vertices into two kinds: *node-vertices* and *wire-vertices*. One way of viewing these is that, in translating the diagrams we saw in chapter 2 into string graphs, node-vertices correspond to the nodes of those diagrams, while wire-vertices "hold the wires in place". For example, the diagram

could be represented as

where the larger white circle is a node-vertex and the smaller black ones are wire-vertices.

Just as terms were defined relative to a signature, we will define string graphs relative to a compressed monoidal signature. We will use a typegraph, so that the type of each node-vertex indicates which morphism symbol it represents, and the type of each wire-vertex indicates which object symbol it represents.

The following is adapted from the definition of a derived typegraph ([23], pp 88).

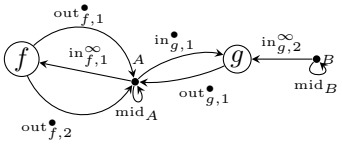
Definition 3.3.5. Given a compressed monoidal signature $T = (O, M, \mathrm{dom}, \mathrm{cod})$, the *derived compressed typegraph* $\mathcal{G}_T$ has vertices $O + M$, a self-loop mid_X for every $X \in O$ and, for every $f \in M$,

- an edge $\mathrm{in}^a_{f,i}$ from X to f, where $(X, a) = \mathrm{dom}(f)[i]$, for each index i of the list $\mathrm{dom}(f)$, and
- an edge $\mathrm{out}^a_{f,i}$ from f to X, where $(X, a) = \mathrm{cod}(f)[i]$, for each index i of the list $\mathrm{cod}(f)$.

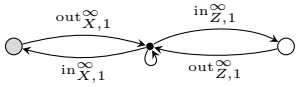
Example 3.3.6. Recall the compressed monoidal signature from example 3.2.5:

$$f : [A^\infty] \to [A^\bullet, A^\bullet]$$
$$g : [A^\bullet, B^\infty] \to [A^\bullet]$$

The derived compressed typegraph would then be



The typegraph for the (spider) Z/X calculus (section 2.2) is very simple:



Let $T = (O, M, \mathrm{dom}, \mathrm{cod})$ be a compressed monoidal signature. Then for a graph (G, τ) in $\mathbf{Graph}/\mathcal{G}_T$ and a vertex v in G, if $\tau(v)$ is in O, we call v a *wire-vertex*. Otherwise, $\tau(v)$ must be in M, and we call v a *node-vertex*. We denote the set of wire-vertices $W(G)$ and the set of node-vertices $N(G)$. If v is a node-vertex and e is an edge incident to v, we call e *variable-arity* if $\tau(e)$ is ∞-tagged (eg: $\mathrm{in}^\infty_{f,1}$) and *fixed-arity* if it is $\bullet$-tagged (eg: $\mathrm{in}^\bullet_{g,1}$). The *fixed edge neighbourhood* $N^\bullet(v)$ of a node-vertex v is the set of edges in the edge neighbourhood of v that are fixed-arity.

In general, we will depict wire-vertices as small black dots.

Definition 3.3.7 (Arity-matching)**.** A map f between $\mathcal{G}_T$-typed graphs G and H is *arity-matching* if for every $v \in N(G)$, the restriction of f to the fixed edge neighbourhood of v is a bijection onto the fixed edge neighbourhood of $f(v)$.

Note that, by considering $\mathcal{G}_T$ as the typed graph $(\mathcal{G}_T, 1_{\mathcal{G}_T})$, we can view typing morphisms as morphisms of $\mathbf{Graph}/\mathcal{G}_T$, and hence apply the above terminology of arity-matching to typing morphisms.

The following is adapted from the definition of a string graph in [23], pp 89; it is equivalent to the original definition if $\mathcal{G}_T$ has no variable-arity edges.

Definition 3.3.8 (String Graph)**.** A $\mathcal{G}_T$-typed graph $(G, \tau_G) \in \mathbf{Graph}/\mathcal{G}_T$ is a *string graph* if τ_G is arity-matching and each wire-vertex in G has at most one incoming edge and at most one outgoing edge. The category $\mathbf{SGraph}_T$ is the full subcategory of $\mathbf{Graph}/\mathcal{G}_T$ whose objects are string graphs.

We refer to a wire-vertex of a string graph G with no incoming edges as an *input*, and write the set of all inputs $\mathrm{In}(G)$. Similarly, a wire-vertex with no outgoing edges is called an *output*, and the set of all such vertices is written $\mathrm{Out}(G)$. The inputs and outputs together form the *boundary* of the graph, written $\mathrm{Bound}(G)$.

We will only consider finite string graphs and finite graphs in $\mathbf{SGraph}_T$, since we are only interested in graphs that are tractable for computers.

Lemma 3.3.9. *If $f : G \to H$ is a morphism in* **SGraph** *and v is a vertex in G that maps to an input (respectively output) of H under f, then v must be an input (respectively output) of G.*

Proof. Suppose e is an edge of G such that $t_G(e) = v$. Then $t_H(f(e))$ must be $f(v)$, and so $f(v)$ cannot be an input unless v is. The output case is symmetric. □

It is worth noting that not every subgraph of a string graph (in $\mathbf{Graph}/\mathcal{G}_T$) is a string graph. In particular, if G is a string graph, n is a node-vertex in G and e is a fixed-arity edge incident to n, a subgraph of G that contains n but not e will fail to satisfy the arity-matching requirement of the typing morphism. However, the intersection and union of two string graphs are still string graphs.

Proposition 3.3.10. *Let G and H, both string graphs, be subgraphs of the string graph K. Then $G \cap H$ and $G \cup H$ are both string graphs.*

Proof. Any subgraph of a string graph satisfies the requirements about wire-vertices, so we only have to consider whether the typing morphisms are arity-matching. What is more, the typing morphism of any subgraph of a string graph must be injective at the fixed-arity neighbourhood of any of its node-vertices; we just need to consider sujectivity.

Let n be a node-vertex in $G \cap H$ and let $\tau^n_{G\cap H}$ be the restriction of $\tau_{G\cap H}$ to the fixed-arity neighbourhood of n. Let e be a fixed-arity edge adjacent to $\tau_{G\cap H}(n)$ in the typegraph. Then both τ^n_G and τ^n_H must have e in their images, and hence so much $\tau^n_{G\cap H}$. Thus $\tau^n_{G\cap H}$ is surjective onto the fixed-arity neighbourhood of the typegraph, and so $\tau_{G\cap H}$ is arity-preserving.

The argument for $G \cup H$ is similar. □

Remark 3.3.11. The construction of the typegraph $\mathcal{G}_T$ means that there can be no edges between node-vertices; any connection between node-vertices is mediated by wire-vertices. Since wire-vertices in string graphs have at most one input and one output, every string graph G is simple, in the sense that for any two vertices v, w of G, there is at most one edge between v and w in each direction.

Proposition 3.3.12. *Every morphism in* $\mathbf{SGraph}_T$ *is arity-matching.*

Proof. Let G and H be string graphs and $f : G \to H$ a graph morphism. Consider $v \in N(G)$. Since $\tau_G = \tau_H \circ f$, we know that the restriction of these maps to $N^\bullet(v)$ is identical. But τ_G is arity-matching, and so its restriction to $N^\bullet(v)$ is a bijection, and hence the same is true of $\tau_H \circ f$. But that means that f restricted to $N^\bullet(v)$ must also be a bijection, as required. □

The following is adapted from a similar proof in [23], pp 90. We can conclude from it that $\mathbf{SGraph}_T$ is what Kissinger calls a *partial adhesive category*; we have chosen not to use this notion on the basis that it does not improve the clarity of the work.

Proposition 3.3.13. *A morphism of* $\mathbf{SGraph}_T$ *is monic if and only if it is injective.*

Proof. Since this holds in $\mathbf{Graph}/\mathcal{G}_T$, any injective map in $\mathbf{SGraph}_T$ must be monic in $\mathbf{Graph}/\mathcal{G}_T$, and hence also monic in $\mathbf{SGraph}_T$.

Suppose we have a non-injective morphism $f : G \to H$ in $\mathbf{SGraph}_T$. Then f must either map two or more edges in G to the same edge in H or map two or more vertices in G to the same vertex in H. In fact, since G and H are both simple, f must map two or more vertices v_i in G to a single vertex v in H. Consider the subgraph K of H containing $f(v_i)$. This must be the vertex, together with any incident fixed-arity edges (and the wire-vertices at the other end of those edges). Then for each v_i, we construct the string graph morphism g_i that takes the single vertex in K to v_i, and the fixed-arity edges to the correct incident edges of v_i (to make the typing morphisms commute). Now all the $f \circ g_i$ are the same morphism, but the g_i morphisms are distinct, so f is not monic. □

For the rest of this chapter, we fix an arbitrary compressed monoidal signature T, and objects and morphisms will be implicitly drawn from $\mathbf{SGraph}_T$ unless otherwise stated.

Wires

It will sometimes be useful to have a different view of a string graph, where we treat each connected string of edges and wire-vertices as a single unit. As the name *wire-vertex* suggests, we will refer to these as *wires*.

We use a different definition of wires (and wire homeomorphisms) to [23]; our definition has several advantages, notably allowing for proposition 3.3.17 and being able to refer to explicit wire homeomorphisms. However, despite the different approach, the two definitions are morally equivalent.

Definition 3.3.14 (Wire)**.** A *wire path* of a string graph G is a path in G containing at least one edge, whose internal vertices are all wire-vertices, and which is not contained in a longer path of this kind.

A *closed wire* of G is a pair (W_V, W_E) such that there is a wire path W of G where W_E is the set of edges in W and W_V is the set of vertices of W.

An *open wire* of G is a pair (W_V, W_E) such that there is a wire path W of G where W_E is the set of edges in W and W_V is the set of internal vertices of W, plus the start/end vertex if W is a cycle.

Wires(G) is the set of all open wires of a string graph G.

Note that an isolated wire-vertex (with no incident edges) is *not* a wire path.

Proposition 3.3.15. *Let G be a string graph. Then two distinct wire paths either describe the same cycle or overlap only on their start and end vertices.*

Proof. The maximality requirement of wire paths means that, except in the case of cycles, the start or end vertex of one path cannot be an internal vertex of another. So distinct paths with a shared component must diverge at some point. However, the requirements of a string graph mean this cannot happen at a wire-vertex, and so can only happen at the start or end vertex. Thus either the paths are identical (or, at least, describe the same cycle) or are distinct everywhere except the start and end vertices. □

This means there is a one-to-one correspondence between open and closed wires. In most cases, it will not matter whether we are referring to an open or a closed wire, and so we will simply use the term *wire*. It also means that each wire path gives rise to only one closed wire and one open wire, and only wire paths that describe the same cycle produce the same wire.

Proposition 3.3.16. *The wire paths, and hence wires, of a string graph G can be sorted into the following disjoint kinds, based on their start and end vertices:*

- *circles, where the start and end vertices are the same;*
- *interior wires, where the start and end vertices are both node-vertices;*
- *bare wires, where the start and end vertices are distinct wire-vertices, the start in* $\mathrm{In}(G)$ *and the end in* $\mathrm{Out}(G)$*;*
- *input wires, where the start vertex is a wire-vertex in* $\mathrm{In}(G)$ *and the end vertex is a node-vertex; and*
- *output wires, where the start vertex is a node-vertex and the end vertex is a wire-vertex in* $\mathrm{Out}(G)$*.*

Additionally, all the wire-vertices of a wire path have the same type, and this is called the type of the wire path (and of its derived open and closed wires).

Proof. The main thing we need to note for the categorisation is that if the path is not a cycle (and hence not a circle), the maximality requirement on a wire path requires the start or end vertex to be in $\mathrm{Bound}(G)$ if it is a wire-vertex, since otherwise it could be an internal vertex of a longer wire path.

The second part is a consequence of the construction of $\mathcal{G}_T$, which disallows edges between wire-vertices of distinct types. □

For wires other than circles, we call the start vertex of the path it is derived from the *source* of the wire, and the end vertex its *target*. Closed wires include their source and target, while open wires do not.

Proposition 3.3.17. *Let G be a string graph. Then $N(G)$, $\text{Bound}(G)$ and the elements of $\text{Wires}(G)$ cover G exactly (with no overlap).*

Proof. Suppose v is a wire-vertex of G not in $\text{Bound}(G)$. Then it must have both an incoming and an outgoing edge. These, together with their respective source and target, form a path whose internal vertices are wire-vertices. This must either be a wire path of G or else be part of a longer wire path, and hence v is part of an open wire of G. Similarly, any edge of G trivially forms a path whose internal vertices are wire-vertices, and is therefore contained in an open wire of G. So these components cover G.

$N(G)$ and $\text{Bound}(G)$ cannot overlap, due to the typegraph. Similarly, open wires contain no node-vertices, and so cannot overlap $N(G)$. Any vertex in an open wire is an internal vertex of a path, and so cannot be in $\text{Bound}(G)$. Finally, proposition 3.3.15 gives us that no two open wires intersect. □

Wire homeomorphisms provide a way to map between graphs that are equivalent under this view.

Definition 3.3.18 (Wire Homeomorphism)**.** A *wire homeomorphism* $f : G \sim H$ between two string graphs G and H consists of three bijective type-preserving functions

- $f_N : N(G) \leftrightarrow N(H)$
- $f_B : \text{Bound}(G) \leftrightarrow \text{Bound}(H)$
- $f_W : \text{Wires}(G) \leftrightarrow \text{Wires}(H)$

such that for each wire w in $\text{Wires}(G)$ with a source v in $N(G)$ (resp. $\text{Bound}(G)$), the source of $f_W(w)$ in H is $f_N(v)$ (resp. $f_B(v)$), and similarly for targets of wires.

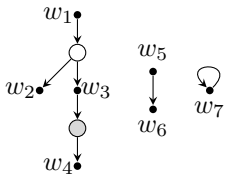
If there is a wire homeomorphism from G to H, then G and H are said to be *wire homeomorphic*.

3.4 Valuation

In order for reasoning using string graphs to be useful, we need some way to interpret them. Node-vertices are intended to correspond to morphisms of a category, wire-vertices to objects of the category and edges to some concept of "information flow". The inputs of a graph naturally correspond in some way to the domain of the morphism represented by the graph, and the outputs to the codomain.

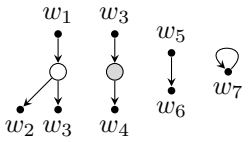
The construction of the value of a graph is essentially the same as the one Kissinger uses when he constructs a free traced SMC and a free compact closed category using string graphs ([23], pp 96-99, 104-112). We bring the notion of the value of a graph to the fore here, however, as we are more concerned with that than with free categories.

Our approach will be to fix an order on the inputs and outputs of a graph, using *framed cospans*, and then to break down the graph into *elementary subgraphs* that we can use a valuation (definition 3.2.7) to assign values (morphisms) to. For example, the graph

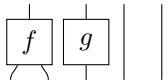


(3.1)

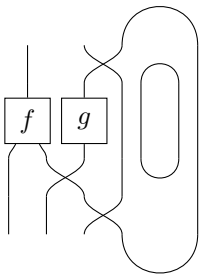
will be broken down into



which will be assigned morphisms, say f, g, 1_A and 1_B respectively. We then take the tensor product of these morphisms



and use the trace, in the form of a *contraction*, to link the morphisms together in the manner indicated by the connections of the original string graph



Swap maps and dual maps are then applied as necessary to get the inputs and outputs in the right order.

3.4.1 Framed String Graphs

Order is important. In a monoidal category, the morphisms

$$f : A \otimes B \to C \otimes D \qquad \text{and} \qquad g : B \otimes A \to D \otimes C$$

cannot be the same, although in a symmetric monoidal category, it is possible that

$$f = \gamma_{D,C} \circ g \circ \gamma_{A,B}$$

In particular, consider the swap and identity morphisms for $A \otimes B$ in a symmetric monoidal category

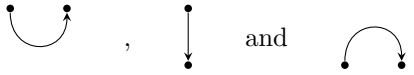
$$\gamma_{A,B} : A \otimes B \to B \otimes A \qquad \text{and} \qquad 1_{A \otimes B} : A \otimes B \to A \otimes B$$

This would naturally be represented in string graph form as

However, these are the *same graph* (or, at least, isomorphic), and so there is no sensible way to assign one the value of $\gamma_{A,B}$ and the other $1_{A \otimes B}$. In a compact closed category, we have a similar issue with the morphisms

$$d_A : A \otimes A^* \to I \qquad , \qquad 1_A : A \to A \qquad \text{and} \qquad e_A : I \to A^* \otimes A$$

whose graph representations should look like



To deal with this, we use *framed cospans*[23].

Definition 3.4.1 (Framed Cospan)**.** A *string graph frame* is a triple $(X, <, \text{sgn})$ where X is a string graph consisting only of isolated wire-vertices, $<$ is a total order on V_X, the vertices of X, and $\text{sgn} : V_X \to \{+, -\}$ is the *signing map*.

A cospan $X \xrightarrow{d} G \xleftarrow{c} Y$ is called a *framed cospan* if

1. X and Y are string graph frames
2. G contains no isolated wire-vertices
3. the induced map $[d, c] : X + Y \to G$ restricts to an isomorphism $[d, c]' : X + Y \cong \text{Bound}(G)$
4. for every $v \in V_X$, $d(v) \in \text{In}(G) \Leftrightarrow \text{sgn}(v) = +$
5. for every $v \in V_Y$, $c(v) \in \text{Out}(G) \Leftrightarrow \text{sgn}(v) = +$

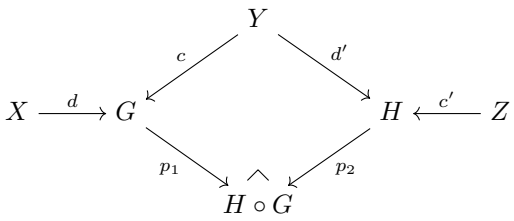
The frames can be seen as "holding the boundary in place". The signing map indicates inputs that appear in the codomain of the span, and outputs that appear in the domain, with a $-$. These correspond to dual objects in compact closed categories. A framed cospan where every vertex in both frames is $+$ is called *positive*.

Given a string graph G, we will often use $\hat{G}$ to refer to some fixed *framing* of it (ie: a framed cospan where the shared domain is G). The particular framing, if it is relevant, will be given by the context.

The composition $\hat{H} \circ \hat{G}$ of two framed cospans

$$X \xrightarrow{d} G \xleftarrow{c} Y \qquad Y \xrightarrow{d'} H \xleftarrow{c'} Z$$

is formed by the pushout



where the resulting cospan is

$$X \xrightarrow{p_1 \circ d} H \circ G \xleftarrow{p_2 \circ c'} Z$$

Remark 3.4.2. We remarked earlier that proposition 3.3.12 means that $\mathbf{SGraph}_T$ is what Kissinger calls a partial adhesive category in [23]. This, and the value construction above, form the necessary components for applying his proofs ([23], pp 104-112) that that classes of framed cospans up to wire homeomorphism form a compact closed category, and homeomorphism classes of positive framed cospans form a symmetric traced category, to our extended version of string graphs.

3.4.2 Indexings and Contractions

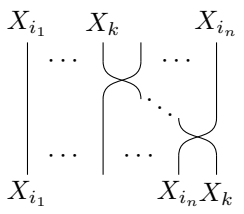
Before we define the value of a graph, we will need some notation. We fix a traced symmetric monoidal category $\mathcal{C}$.

Definition 3.4.3 (Indexing)**.** Let O be a (small) set of objects of $\mathcal{C}$. For an object X of $\mathcal{C}$, an *X-indexing* is an indexed tensor product of the form

$$X_{i_1} \otimes \ldots \otimes X_{i_n}$$

that is isomorphic to X. We call a morphism $f : X \to Y$ *indexed* when we have fixed an X-indexing and a Y-indexing.

For an X-indexing as above, we define the map $\text{shunt}_{X:k}$ to be the (unique) isomorphism constructed from the identity and swap maps of $\mathcal{C}$ that moves the X_k element of the X-indexing to the end and leaves all others the same, as in the following diagram:



Definition 3.4.4 (Contraction)**.** Given an indexed map $f : X_{i_1} \otimes \ldots \otimes X_{i_m} \to Y_{j_1} \otimes \ldots \otimes Y_{j_n}$, where $X_k = Y_l$, the *contraction of f from k to l*, $C_k^l(f)$, is the morphism

$$\mathrm{Tr}^{X_k}(\mathrm{shunt}_{Y:l} \circ f \circ \mathrm{shunt}^{-1}_{X:k})$$

A contraction of an indexed morphism yields another indexed morphism. We can therefore apply multiple contractions to the same morphism, and these contractions commute:

Proposition 3.4.5 (Kissinger)**.** *For an indexed morphism f and distinct indices i ,i', j and j',*

$$C_i^j(C_{i'}^{j'}(f)) = C_{i'}^{j'}(C_i^j(f))$$

Proposition 3.4.6. *Given X- and Y-indexings $X_{i_1} \otimes \ldots \otimes X_{i_m}$ and $Y_{j_1} \otimes \ldots \otimes Y_{j_n}$, and morphisms*

$$f : X_{i_1} \otimes \ldots \otimes X_{i_k} \to Y_{j_1} \otimes \ldots \otimes Y_{j_l}$$

and

$$g : X_{i_{k+1}} \otimes \ldots \otimes X_{i_m} \to Y_{j_{l+1}} \otimes \ldots \otimes Y_{j_n}$$

if $p = i_a$ for some $a > k$ and $q = j_b$ for some $b > l$, then

$$C_p^q(f \otimes g) = f \otimes C_p^q(g)$$

Proof. This follows from the definition and the laws of traces. □

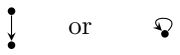
3.4.3 Value

Let $\mathcal{V}$ be a symmetric traced category, and let v be an expansion-order invariant valuation of T over $\mathcal{V}$. For a framed cospan $\hat{G} = X \to G \leftarrow Y$ of $\mathbf{SGraph}_T$, we define the *value* of $\hat{G}$ under v in the following manner.

The *elementary subgraphs* of G are those subgraphs that consist of one of the following:

- a single node-vertex together with its incident edges and adjacent wire-vertices

- an edge between wire-vertices, together with those wire-vertices (which may be the same vertex)

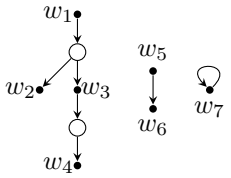


- a wire-vertex that has no incident edges in G

Note that these are all valid string graphs. The definition of a string graph (and, in particular, the restriction on incident edges of wire-vertices) means that:

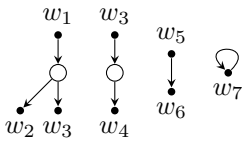
- G is covered by its elementary subgraphs,
- no node-vertex or edge appears in more than one elementary subgraph,
- wire-vertices appear in at most two elementary subgraphs, and
- any wire-vertex that does appear in two elementary subgraphs is an input in one and an output in the other.

Example 3.4.7. The string graph



(3.2)

has the following four elementary subgraphs:



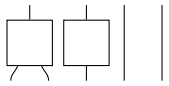
We then assign a value to each of these subgraphs, as well as a wire-vertex-based indexing for the domain and codomain of that value.

For an elementary subgraph H of G with a node-vertex n, let $f = \tau_H(n)$, and for each incoming edge $\text{in}^a_{f,i}$ of f in $\mathcal{G}_T$, let $d(i)$ be the size of the preimage of the edge under τ_H (ie: the number of edges adjacent to n that map to it), and define $c(j)$ similarly for each outgoing edge $\text{out}^a_{f,j}$. Then these are arity counts for f, since τ_G, and hence τ_H, is arity-matching. The value of H is then $v^f_m(d, c)$.

Consider A, the domain of $v^f_m(d, c)$, and let $D(H)$ be the set of wire-vertices of H that are the source of an incoming edge of n. If $D(H)$ is empty, $A = I$. Otherwise, for each input w of H with adjacent edge e, let $\lambda(w) = i$, where $\tau_H(e) = \text{in}^a_{f,i}$. We can then choose a total order for $D(H)$ such that λ is non-decreasing, and this ordered set provides an A-indexing with the property that if $\text{dom}(f) = [\alpha_1, \ldots, \alpha_m]$, then $A_w = \alpha_{\lambda(w)}$. We can construct a similar indexing for the codomain of $v^f_m(d, c)$ using $C(H)$, the set of targets of edges whose source is n (unless this is empty).

If H is instead an edge e connecting two wire-vertices, we assign the value $1_{v_o(Z)} : v_o(Z) \to v_o(Z)$, where Z is the image of $s(e)$ (and hence also of $t(e)$). We set $D(H) = \{s(e)\}$ and $C(H) = \{t(e)\}$ (note that these may be the same). If H is an isolated wire-vertex w of type Z, we similarly assign the value $1_{v_o(Z)}$ and set $D(H) = C(H) = \{w\}$.

We then let g_0 be the tensor product of the values of each elementary subgraph of G. A diagrammatic representation of this morphism for the graph (3.2) is

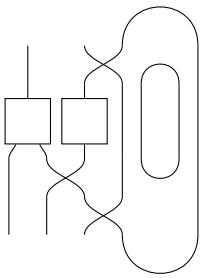


Let $W_1(G)$ be the union of $D(H)$ and $W_2(G)$ the union of $C(H)$ for the elementary subgraphs H of G. The total orders we placed on the inputs of each elementary subgraph of G, together with the order of the tensor product g_0, gives us an order on $W_1(G)$, and (providing $W_1(G)$ is not empty) this gives rise to a $W_1(G)$-based indexing of the domain of g_0 that agrees with the indexings we constructed for the elementary subgraphs; the same can be done for the codomain of g_0 using $W_2(G)$.

Now we order the wire-vertices $w_1, \ldots, w_m$ of G that appear in both $W_1(G)$ and $W_2(G)$, but are not isolated wire-vertices in G, and for $1 \leq i \leq m$ let

$$g_i = C_{w_i,w_i}(g_{i-1})$$

For the previous example, this would result in



Note that $\mathrm{In}(G) \setminus \mathrm{Out}(G) = W_1(G) \setminus W_2(G)$ and, conversely, $\mathrm{Out}(G) \setminus \mathrm{In}(G) = W_2(G) \setminus W_1(G)$. This means that, given that the isolated wire-vertices of G are the only vertices to be both inputs and outputs of G, the remaining indices in the indexing of the domain of g_m are exactly the inputs of G, and those in the indexing of the codomain are its outputs.

This means we can simply precompose and postcompose g_m with the necessary swap maps to re-order the indexings of its domain and codomain to agree with the orderings of the frames of $\hat{G}$. We call the resulting morphism $v(\hat{G})$, the value of $\hat{G}$.

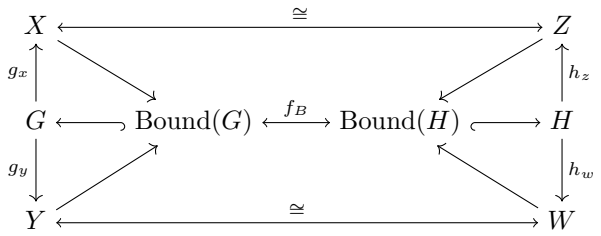
This final reordering means that the choice of order of elementary subgraphs when constructing g_0 will not affect the value of $\hat{G}$. The fact that contractions commute means that the choice of order of wire-vertices in G when constructing the maps $g_1, \ldots, g_m$ does not affect the value of $\hat{G}$. Finally, since the valuation v is expansion-order invariant, the choice of indexing for the domains and codomains of the elementary subgraphs of G does not affect the value of $\hat{G}$.

In a compact closed category, a combination of swap maps and the maps associated with dual objects can be used to produce $v(\hat{G})$ from g_m.

Note that the domain and codomain of $v(\hat{G})$ depend only on the frames, not on G. Note also that the frames only affect the final part of the construction, and so the values of two different framed cospans with the same internal string graph are the same up to pre- and postcomposition with swap maps if both are positive, and swap maps and dual object maps otherwise.

We will assume all valuations are expansion-order invariant from now on.

Proposition 3.4.8. *Suppose $\hat{G} = X \xleftarrow{g_x} G \xrightarrow{g_y} Y$ and $\hat{H} = Z \xleftarrow{h_z} H \xrightarrow{h_w} W$ are framings and $f : G \to H$ is a wire homeomorphism such that the following diagram commutes*



where the triangles on the left and right are taken from the definition of a cospan frame, and the top and bottom morphisms are order- and sign-preserving. Then, for every valuation v,

$$v(\hat{G}) = v(\hat{H})$$

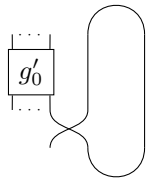
Proof. We start by noting that the diagram above means we can identify X with Z, Y with W and Bound(G) with Bound(H) in a consistent manner.

Now we make the observation that if G and H are wire-homeomorphic, then there is a third graph K that is wire-homeomorphic to both where no two wire-vertices are adjacent to each other. What is more, K can be reached from both G and H by a (finite) series of 0 or more *wire contractions*, where an edge e between two distinct wire-vertices is removed, and $s(e)$ and $t(e)$ are identified. Thus we just have to show that the value of a framed cospan is stable under wire contraction, and the result follows by transitivity.

So let e be an edge of G whose source and target are distinct wire-vertices. Then e (together with its source and target) is an elementary subgraph of G. If $s(e)$ is an input of G and $t(e)$ is an output, contracting e will make no difference to the value of $\hat{G}$, as we will simply be replacing one elementary subgraph with another that has the same value, and there are no contractions in either case.

So suppose $s(e)$ is *not* an input of G. Then $s(e) \in W_2(G)$, and there is a contraction $C^{s(e)}_{s(e)}$. We can choose to place the elementary subgraph containing e last and apply the contraction on its source

first. $C^{s(e)}_{s(e)}(g_0)$ will then have the form (where we only display the last swap of the contraction)



where g_0' is the tensor product of the elementary subgraphs of the contracted form of G together with all but one swap from the contraction of $s(e)$. Then we have

$$C^{s(e)}_{s(e)}(g_0) = g_0'$$

by the trace axioms. The extra swaps are irrelevant to the value of the contracted form of $\hat{G}$ as they are either absorbed into $C^{t(e)}_{t(e)}$ if $t(e)$ is not an output, or are corrected at the end to account for the frame ordering.

The case where $t(e)$ is not an output of G is symmetric. So the value of G remains the same after contracting e, as required, and hence we have the required result. □

3.5 Graph Equations

Our aim is to be able to mechanise equational reasoning for diagrams using string graphs. However, we first need to establish what we even mean by a "graph equation", and develop an equational logic for this concept.

Recall that in the equational logic for terms, there are rules that state that $\approx$ is closed under contexts and substitutions. We can view substitutions as a sort of internal context. For example, if we say

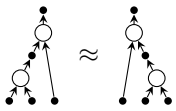
$$f(x) \approx g(x)$$

we mean that we can replace f with g (or vice versa) regardless of what is either inside or surrounding that symbol. This approach is well-suited to the tree-like structure of terms, where there is a single external context and variables provide an easy way of labelling internal contexts, but does not work so well for graphs, which have only an external context, but may have multiple points of interaction with that context.

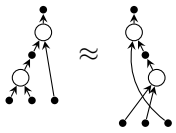
Declaring that two morphisms are equal only makes sense if both morphisms have the same type. Given our intended interpretation of the string graphs, this means that they need the same inputs and outputs (respecting types). But that is not enough; consider the associativity law for multiplication. For terms, we would have something like

$$m(m(x, y), z) \approx m(x, m(y, z))$$

while with string graphs we would want something like



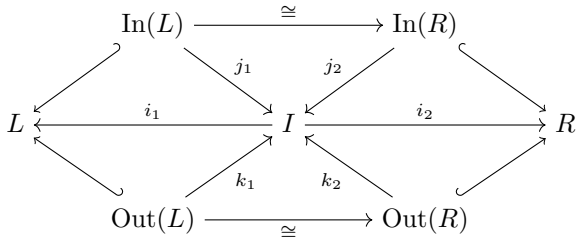
Visually, the meaning of this is clear, but we need some way to encode the correspondance between inputs; after all, one way of correlating the inputs would result in a law that was simply a consequence of commutativity:



We encode the correspondance of inputs and outputs using a span of graph morphisms whose images are the boundaries of the two graphs. We place some extra coherence requirements on the span to ensure that inputs are linked to inputs and outputs to outputs.

Definition 3.5.1 (String Graph Equation; [23], pp 93)**.** A *string graph equation* $L \approx_{i_1,i_2} R$ is a span $L \xleftarrow{i_1} I \xrightarrow{i_2} R$ where:

- L and R contain no isolated wire-vertices;
- $\mathrm{In}(L) \cong \mathrm{In}(R)$ and $\mathrm{Out}(L) \cong \mathrm{Out}(R)$;
- $\mathrm{Bound}(L) \cong I \cong \mathrm{Bound}(R)$; and
- the following diagram commutes, where j_1, j_2, k_1 and k_2 are the coproduct inclusions into the boundary composed with the above isomorphisms:



We can assemble similar laws to those we had in the term-based equational logic. Axiom, reflexivity and symmetry are straightforward:

$$(\textsc{Axiom})\ \frac{G \approx_{i,j} H \in E}{E \vdash G \approx_{i,j} H} \qquad (\textsc{Refl})\ \frac{}{E \vdash G \approx_{b_G,b_G} G} \qquad (\textsc{Sym})\ \frac{E \vdash G \approx_{i,j} H}{E \vdash H \approx_{j,i} G}$$

where $b_G : \mathrm{Bound}(G) \hookrightarrow G$ is the obvious inclusion of the boundary of G into G. Transitivity requires a bit more work to express. If we have the graph equations

$$G \xleftarrow{i} I \xrightarrow{j} H$$

and

$$H \xleftarrow{k} J \xrightarrow{l} K$$

then the constraints on graph equations mean that $k^{-1} \circ j$ is both well-defined and an isomorphism $I \cong J$. If we take $p := i$ and $q := l \circ k^{-1} \circ j$, we can express transitivity as

$$(\textsc{Trans}) \quad \frac{E \vdash G \approx_{i,j} H \qquad E \vdash H \approx_{k,l} K}{E \vdash G \approx_{p,q} K}$$

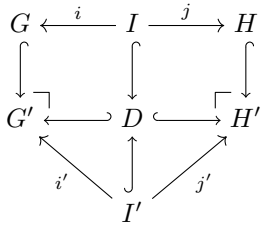
For this to reflect intuitive equational reasoning, though, we need to be able to apply equations in the context of larger graphs. Given graph equations

$$G \xleftarrow{i} I \xrightarrow{j} H$$

and

$$G' \xleftarrow{i'} I' \xrightarrow{j'} H'$$

such that there are monomorphisms making the following diagram commute, and the squares in it pushouts,



then we have the following rule

$$(\textsc{Leibniz}) \quad \frac{E \vdash G \approx_{i,j} H}{E \vdash G' \approx_{i',j'} H'}$$

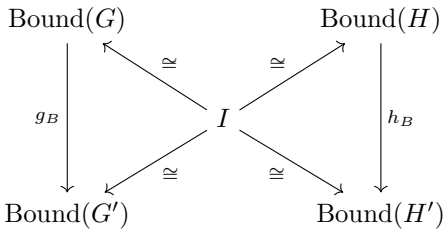
We actually need one more rule, to account for wire homeomorphisms. Given graph equations

$$G \xleftarrow{i} I \xrightarrow{j} H$$

and

$$G' \xleftarrow{i'} I \xrightarrow{j'} H'$$

we say they are wire homeomorphic if there are wire homeomorphisms $g : G \sim G'$ and $h : H \sim H'$ such that



(3.3)

where the isomorphisms are those induced by i, j, i' and j' (see definition 3.5.1). Given two such wire homeomorphic equations, we have the following rule

$$(\textsc{Homeo})\ \frac{E \vdash G \approx_{i,j} H}{E \vdash G' \approx_{i',j'} H'}$$

We should note here that multiple stacked applications of the Leibniz rule can be condensed into a single application.

Proposition 3.5.2. *Suppose we have the proof tree fragment*

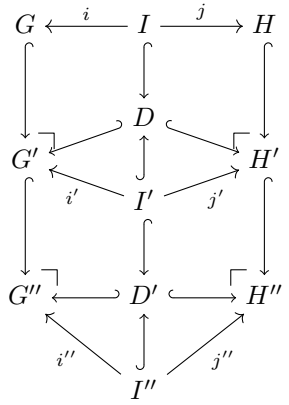
$$\textsc{Leibniz}\ \frac{\textsc{Leibniz}\ \dfrac{E \vdash G \approx_{i,j} H}{E \vdash G' \approx_{i',j'} H'}}{E \vdash G'' \approx_{i'',j''} H''}$$

Then

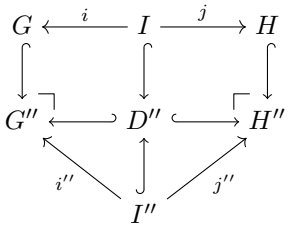
$$\textsc{Leibniz}\ \frac{E \vdash G \approx_{i,j} H}{E \vdash G'' \approx_{i'',j''} H''}$$

is a valid proof tree fragment.

Proof. We must have the following commuting diagram



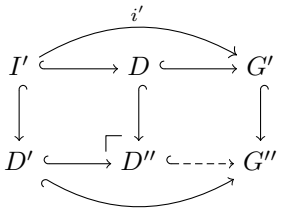
We just need to find a string graph D'' and suitable monomorphisms to make



(3.4)

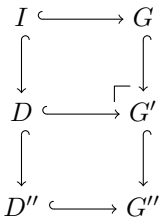
commute. We will only demonstrate how to construct the left part of the diagram, as the right side can be constructed in the same manner. First of all, D'' and its inclusion into G'' are formed by

pushout:



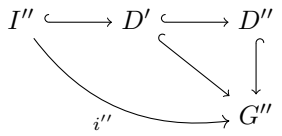
What is more, because the left square and outer squares are both pushouts, the right square must also be a pushout.

Now the left square of (3.4) can easily be seen to commute:



What is more, we have just established that the bottom square is a pushout, so the outer square must also be a pushout.

The triangle between I'', D'' and G'' follows easily:



□

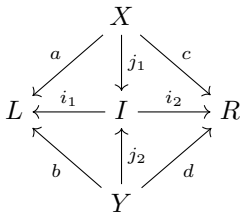
3.5.1 Soundness

We want to make sure these rules are *sound* with respect to the valuation we described in section 3.4. In other words, if the assumptions of one of the preceding rules holds under a particular valuation, then the conclusion must hold under the same valuation.

However, the rules above deal with string graph equations, and valuations apply to framed cospans. We need to introduce frames to both sides of a string graph equation in a way that is compatible with the span that forms the equation.

Definition 3.5.3. A *framing* of a string graph equation $L \approx_{i_1,i_2} R$ is a pair of framed cospans $X \xrightarrow{a} L \xleftarrow{b} Y$ and $X \xrightarrow{c} R \xleftarrow{d} Y$ such that there are morphisms j_1 and j_2 forming the coproduct (ie:

disjoint union) $X \xrightarrow{j_1} I \xleftarrow{j_2} Y$ and making the following diagram commute:



Note that it is always possible to frame a string graph equation in the following way: let X be a subgraph of I, and Y its complement. Place an arbitrary ordering on X and Y, and construct signing maps so that $sgn_X(x) = +$ if and only if $i_1(x)$ (and hence also $i_2(x)$) is an input, and $sgn_Y(y) = +$ if and only if $i_1(y)$ (and hence also $i_2(y)$) is an output. X and Y are then frames. We set j_1 and j_2 to be the natural inclusions, and define a, b, c and d by the above diagram. $X \xrightarrow{a} L \xleftarrow{b} Y$ and $X \xrightarrow{c} R \xleftarrow{d} Y$ are then framed cospans, and so we have a framing of the string graph equation.

In addition, fixing a framed cospan for either L or R fixes a framing for the equation. This is because fixing a and b, for example, fixes j_1 and j_2 (since i_1 is monic).

We say that a string graph equation $L \approx_{i_1,i_2} R$ holds under a valuation v if for all framings $(\hat{L}, \hat{R})$ (all positive framings if the valuation is not over a compact closed category) we have that $v(\hat{L}) = v(\hat{R})$. In fact, this is the same as saying we have this for *any* framing.

Proposition 3.5.4. *For any valuation v and any string graph equation $L \approx_{i_1,i_2} R$, if (L_1, R_1) and (L_2, R_2) are framings of $L \approx_{i_1,i_2} R$ (where both are positive unless v is over a compact closed category), then $v(L_1) = v(R_1)$ if and only if $v(L_2) = v(R_2)$.*

Proof. Consider the domains and codomains of $v(L_1)$ and $v(L_2)$, indexed by the vertices of their respective frames, X_1, Y_1, X_2 and Y_2.

Using the same construction as for the value, we can find a tensor product of identities i and maps c and d composed of identities, swaps and dual maps such that

$$v(L_1) = c \circ (v(L_2) \otimes i) \circ d$$

and this construction depends only on the frames of the cospans, since these are what determine the domains and codomains of the values and their indexings. But this means that we also have

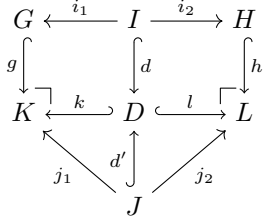
$$v(R_1) = c \circ (v(R_2) \otimes i) \circ d$$

since L_1 and R_1 have the same frames, and L_2 and R_2 also share their frames. Thus if $v(L_2) = v(R_2)$, then $v(L_1) = v(R_1)$.

The same argument works in reverse, giving us that if $v(L_1) = v(R_1)$, then $v(L_2) = v(R_2)$. □

Soundness of the reflexivity and symmetry axioms follows from the reflexivity and symmetry of $=$. For transitivity, we just need to note that, given a framing $(\hat{G}, \hat{K})$ of $G \approx_{p,q} K$, there is a framed cospan $\hat{H}$ such that $(\hat{G}, \hat{H})$ and $(\hat{H}, \hat{K})$ are framings of $H \approx_{k,l} K$ and $G \approx_{i,j} H$ respectively.

Proposition 3.5.5. *The* Leibniz *rule for string graphs is sound. In other words, if we have the following commuting diagram*



where the top and bottom spans are string graph equations, a framing $(\hat{G}, \hat{H})$ of $G \approx_{i_1,i_1} H$ and framing $(\hat{K}, \hat{L})$ of $K \approx_{j_1,j_2} L$, then for all valuations v,

$$v(\hat{G}) = v(\hat{H}) \;\Rightarrow\; v(\hat{K}) = v(\hat{L})$$

Proof. By proposition 3.5.4, it suffices to choose a specific framing for each equation, instead of showing it for all framings.

Since all the arrows in the above diagram are monic (and hence injective), we will consider them to be subgraph relations (so G is a subgraph of K by g, and so on). For convenience, we will use D' to refer to D less any isolated wire-vertices.

We start by noting that I consists only of isolated wire-vertices, but G and K have *no* isolated wire-vertices. This means that, since K is a pushout of D and G from I, each elementary subgraph of K must be an elementary subgraph of exactly one of D' or G. Similarly, $W_1(K)$ (defined in section 3.4.3) is the disjoint union of $W_1(G)$ and $W_1(D')$, and likewise for $W_2(K)$. The same results hold for L.

It then suffices to show that, when calculating the value of a framed cospan $\hat{K}$ of K, we can choose framings $(\hat{G}, \hat{H})$ and $(\hat{K}, \hat{L})$ for the equations such that $v(\hat{K})$ is of the form

$$C_{a_n}^{a_n}(\cdots C_{a_1}^{a_1}(d_0 \otimes v(\hat{G}))\cdots)$$

and $v(\hat{L})$ is

$$C_{a_n}^{a_n}(\cdots C_{a_1}^{a_1}(d_0 \otimes v(\hat{H}))\cdots)$$

for some indexed morphism d_0. The result then follows immediately.

We choose an order for the elementary subgraphs of D' (none of which are isolated wire-vertices), and an order for the variable-arity edges of those subgraphs. Tensoring the elementary subgraphs together in order gives us d_0, which can be indexed by $W_1(D')$ and $W_2(D')$, appropriately ordered. We do the same for G and H to get g_0 and h_0, indexed similarly.

Now $k_0 = d_0 \otimes g_0$ and $l_0 = d_0 \otimes h_0$ are valid choices for the same values for K and L, and we can order $W_1(K)$ using the order of $W_1(D')$ followed by the order of $W_1(G)$, and similarly for $W_2(K)$, $W_1(L)$ and $W_2(L)$. These can then be used to index k_0 and l_0, and these indexings agree with the indexings for d_0, g_0 and h_0.

Next, we arbitrarily order $W_1(G) \cap W_2(G)$ as $b_1, \ldots, b_p$ and $W_1(H) \cap W_2(H)$ as $c_1, \ldots, c_q$. We apply the contractions $C^{b_i}_{b_i}$ in order to k_0 and $C^{c_j}_{c_j}$ to h_0 to get k_p and h_q respectively. At this point, we note that, by proposition 3.4.6,

$$k_p = d_0 \otimes C^{b_p}_{b_p}(\cdots C^{b_1}_{b_1}(g_0)\cdots) = d_0 \otimes g_p$$

and similarly $l_q = d_0 \otimes h_q$. But if we choose the framing $(\hat{G}, \hat{H})$ so that the ordering of the frames matches the ordering of $W_1(G)$ and $W_2(G)$, we have that $v(\hat{G}) = g_p$. To get $v(\hat{H})$, we may need to pre- and post-compose with swap maps to get an order that agrees with the frames. Call these s_1 and s_2. Then $v(\hat{H}) = s_2 \circ h_q \circ s_1$.

If we order $(W_1(K) \cap W_2(K)) \setminus (W_1(G) \cap W_2(G))$ as $a_1, \ldots, a_n$, a similarly suitable choice of the framing $(\hat{K}, \hat{L})$ gives us

$$v(\hat{K}) = C^{a_n}_{a_n}(\cdots C^{a_1}_{a_1}(d_0 \otimes v(\hat{G}))\cdots)$$

It remains to show that

$$v(\hat{L}) = C^{a_n}_{a_n}(\cdots C^{a_1}_{a_1}(d_0 \otimes (s_2 \circ h_q \circ s_1))\cdots)$$

We need to consider the decomposition of s_1 and s_2 into their individual swap operations. If any given swap operates on an index that is then contracted (ie: one of the a_i), it can be subsumed into that contraction, due to the way contractions are defined. So we can construct reordering maps s_1' and s_2' that only operate on inputs and outputs (respectively) of L such that

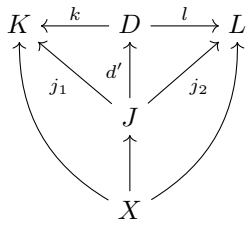
$$C^{a_n}_{a_n}(\cdots C^{a_1}_{a_1}(d_0 \otimes (s_2 \circ h_q \circ s_1))\cdots) = C^{a_n}_{a_n}(\cdots C^{a_1}_{a_1}(d_0 \otimes (s_2' \circ h_q \circ s_1'))\cdots)$$

But this is the same as

$$(1 \otimes s_2') \circ C^{a_n}_{a_n}(\cdots C^{a_1}_{a_1}(d_0 \otimes h_q)\cdots) \circ (1 \otimes s_1')$$

We will refer to this (indexed) morphism as $\lambda : A \to B$.

We need to show that indexing of this morphism agrees with the order on the frames of $\hat{L} = X \hookrightarrow L \hookleftarrow Y$. We will just treat the X frame, as the Y frame is symmetric. Note that we have the following commuting diagram, by the definition of a framing, which allows us to consider X a subgraph of D.



We split X, which consists of the inputs of K (and therefore of L) into two parts: X_1 are those vertices that are inputs of D', and X_2 are the inputs of G (and hence of H). Note that we can consider X_1 a subgraph of D' and X_2 a subgraph of I.

Suppose $a < b \in X$. Then either both are in X_1, both are in X_2 or $a \in X_1$ and $b \in X_2$. In the latter case, we know that a appears before b in the indexing of the domain of λ due to the construction of λ. Suppose both are in X_1. Then we know that A_a appears before A_b in the indexing of d_0, because that is how we chose the ordering on X. Otherwise, both are in X_2. We know that A_a appears before A_b in the indexing of g_0, and hence in the indexing of g_p. If the same is true of the indexing of h_q, it must be the case that s_1, and hence s_1', does not swap the order of A_a and A_b, and so A_a appears before A_b in the indexing of λ. Otherwise, if A_b is before A_a in the indexing of h_q, s_1 (and hence s_1') must swap A_a with A_b, and so we still have that A_a is before A_b in the indexing of λ.

So $v(\hat{L}) = \lambda$, as required. □

Proposition 3.5.6. *The* HOMEO *rule for string graphs is sound. In other words, given wire-homeomorphic string graph equations $G \approx_{i,j} H$ and $G' \approx_{i',j'} H'$ with respective framings $(\hat{G}, \hat{H})$ and $(\hat{G}', \hat{H}')$,*

$$v(\hat{G}) = v(\hat{H}) \;\Rightarrow\; v(\hat{G}') = v(\hat{H}')$$

Proof. The precondition on the wire homeomorphisms allows us to construct framings of the equation that are consistent with each other in the manner required by proposition 3.4.8. From that proposition, we get

$$v(\hat{G}') = v(\hat{G}) \qquad \text{and} \qquad v(\hat{H}) = v(\hat{H}')$$

and it follows by transitivity of $=$ that

$$v(\hat{G}) = v(\hat{H}) \;\Rightarrow\; v(\hat{G}') = v(\hat{H}')$$

□

3.6 Graph Rewriting

Graph rewriting has been around in some form since at least the 60s; it made an early appearance in *graph grammars*, a field that aimed to apply formal language theory to multidimensional objects rather than just linear strings, notably in [33] and [36].

In term rewriting, we replaced a subterm of a term with another term that we viewed as equivalent in some way. In a similar manner, graph rewriting (at least as far as this thesis is concerned) aims to replace a subgraph of a graph with another "equal" graph. There are several approaches to this problem, almost all of which involve a graph L to be searched for and removed from the target graph

G, a graph R to replace it with and some *embedding transformation* E describing how to map from the connectivity of L with $G \setminus L$ to edges between R and $G \setminus L$.

Algebraic graph rewriting systems express E using constructs from category theory, which provides a useful set of tools for reasoning about such systems. Using these systems allows us to make use of established work, such as proofs that independent rewrites (which operate on different parts of G) can be applied in either order and acheive the same result[14].

There are two main approaches to algebraic graph rewriting: single-pushout (SPO) and double-pushout (DPO). DPO was the first algebraic graph rewriting system, introduced by Ehrig, Pfender and Schneider in [15]. A rewrite rule is represented as a span of graphs

$$L \xleftarrow{i} I \xrightarrow{j} R$$

and a rewrite from G to H is a pair of overlapping pushouts

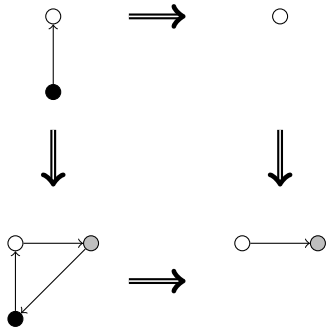
$$\begin{array}{ccccc} L & \xleftarrow{i} & I & \xrightarrow{j} & R \\ \downarrow & & \downarrow & & \downarrow \\ G & \xleftarrow{} & D & \xrightarrow{} & H \end{array}$$

The embedding transformation is described by i, j and I. We can think of I as being the interface of the rewrite rule, the part allowed to interact with the rest of G. This part is held in place when the rest of L is removed from G, allowing the connectivity with $G \setminus L$ to be maintained.

SPO, as presented by Löwe in [41], generalises DPO by using a single pushout

$$\begin{array}{ccc} L & \longrightarrow & R \\ \downarrow & & \downarrow \\ G & \longrightarrow & H \end{array}$$

in the category of partial graph morphisms. This allows for rewrites that would be prohibited under DPO, such as deleting vertices in unknown contexts. For example the SPO rewrite



has no equivalent DPO rewrite, because of the edge between the grey and black vertices.

We will make use of DPO as our graph rewriting system, as we do not require (and, in fact, wish to avoid) the extra flexibility of SPO.

Similar to the term case, we direct a string graph equation (definition 3.5.1) to get a *string graph rewrite rule*. Given a string graph rewrite rule $L \xleftarrow{i} I \xrightarrow{j} R$, a *string graph rewrite* is a pair of pushouts of monomorphisms in $\mathbf{SGraph}_T$

$$\begin{array}{ccccc} L & \xleftarrow{i} & I & \xrightarrow{j} & R \\ {\scriptstyle m}\downarrow & & \downarrow & & \downarrow \\ G & \hookleftarrow & D & \hookrightarrow & H \end{array} \tag{3.5}$$

(which should also be familiar from the rules of section 3.5).

Note that the inclusion of D into G must contain Bound(G) in its image. This is because any input or output of G in the image of m must map from an input or output of L, and must therefore be in the image of i. The same is true of the inclusion of D into H. What is more, the constraints on i and j mean that the preimages of Bound(G) and Bound(H) under these inclusions coincide. The same is also true for In(G) and In(H) and for Out(G) and Out(H). If we let I' be the shared preimage of the boundaries of G and H, then – providing G and H have no isolated wire-vertices – the span at the bottom of

$$\begin{array}{ccccc} G & \hookleftarrow & D & \hookrightarrow & H \\ & {\scriptstyle i'}\nwarrow & \uparrow & \nearrow{\scriptstyle j'} & \\ & & I' & & \end{array}$$

is a string graph rewrite rule.

As the term "rewriting" suggests, what we actually want to do is start with the string graph rewrite rule $L \twoheadrightarrow_{i,j} R$ and G, and find the remainder of the diagram. We start by finding a *matching*, a map $m : L \to G$ (see (3.5)) such that i and m can be extended to a pushout; ie: they have a *pushout complement*.

Definition 3.6.1 (Pushout complement)**.** A *pushout complement* for a pair of arrows

$$A \xrightarrow{f} B \xrightarrow{g} C$$

is an object B' and a pair of arrows

$$A \xrightarrow{f'} B' \xrightarrow{g'} C$$

such that

$$\begin{array}{ccc} A & \xrightarrow{f} & B \\ {\scriptstyle f'}\downarrow & & \downarrow{\scriptstyle g} \\ B' & \xrightarrow[g']{} & C \end{array}$$

is a pushout.

The following result is due to Dixon and Kissinger[11], based on a similar result for adhesive categories in [27].

Proposition 3.6.2. *If a pair of arrows (b, f) in* $\mathbf{SGraph}_T$*, where b is monic, have a pushout complement, it is unique up to isomorphism.*

Definition 3.6.3 (String graph matching)**.** For a string graph G and a string graph rewrite rule $L \xleftarrow{i} I \xrightarrow{j} R$, a monomorphism $m : L \to G$ is called a *string graph matching* if $I \xrightarrow{i} L \xrightarrow{m} G$ has a pushout complement.

We require a stronger form of arity-matching in order to guarantee pushout complements:

Definition 3.6.4. A map f between $\mathcal{G}_T$-typed graphs G and H is a *local isomorphism* if for every $v \in N(G)$, the restriction of f to the edge neighbourhood of v is a bijection onto the edge neighbourhood of $f(v)$.

Proposition 3.6.5 (Kissinger)**.** *Let $L \rightarrowtail R$ be a string graph rewrite rule. Then any monic local isomorphism $m : L \to G$ is a string graph matching.*

So if we find a monomorphism $m : L \to G$, we can assemble the first pushout square of (3.5): since i and m are monic, we can view them as expressing the subgraph relation. Then D is obtained by removing from G everything in the image of m but not in the image of $m \circ i$. But we still need to be able to form the second pushout.

Boundary coherence proves to be the right notion for determining when we can form pushouts in $\mathbf{SGraph}_T$.

Definition 3.6.6. A span of morphisms $G \xleftarrow{g} I \xrightarrow{h} H$ in $\mathbf{SGraph}_T$ is called *boundary-coherent* if:

1. for all $v \in \mathrm{In}(I)$, at least one of $g(v)$ and $h(v)$ is an input; and
2. for all $v \in \mathrm{Out}(I)$, at least one of $g(v)$ and $h(v)$ is an output.

A single morphism $f : G \to H$ is called boundary-coherent if the span $H \xleftarrow{f} G \xrightarrow{f} H$ is.

Theorem 3.6.7 (Kissinger)**.** *A span of morphisms $G \xleftarrow{g} I \xrightarrow{h} H$ in* $\mathbf{SGraph}_T$ *has a pushout if and only if it is boundary-coherent.*

Our definition of a string graph rewrite rule means that the boundary coherence of the maps in the left-hand pushout propagates to the maps on the right, meaning that if we can construct one of the pushouts in (3.5), we can always construct the other one.

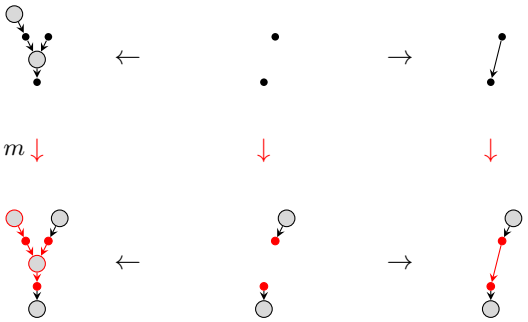
Lemma 3.6.8. *Let $L \xleftarrow{i} I \xrightarrow{j} R$ be a string graph rewrite rule and $f : I \to G$ be a string graph morphism. Then i and f as a span are boundary-coherent if and only if f and j are.*

Proof. This follows directly from the fact that, for all vertices w in I, $i(w)$ is an input (resp. output) of L if and only if $j(w)$ is an input (resp. output) of R. □

Corollary 3.6.9. *Let $L \rightarrowtriangle R$ be a string graph rewrite rule and $m : L \to G$ a monic string graph local isomorphism. Then $L \rightarrowtriangle R$ has an S-rewrite at m.*

Constructing the pushout is easy: pushouts along monomorphisms in **Graph** (and its slice categories) are just graph unions (with an explicit intersection). Likewise, pushout complements from monomorphisms can be calculated using graph subtraction. The above results guarantee that we will get valid string graphs when we do so.

Example 3.6.10. The following string graph rewrite is an application of the string graph rewrite rule for the unit law in the Z/X calculus:



3.6.1 Equational Reasoning with Rewrites

Rewriting is typically done with respect to some set of rewrite rules S, called a *(graph) rewrite system*. We say that a string graph G rewrites to a string graph H under the graph rewrite sytem S (or $G \rightarrowtriangle_S H$) if there is a string graph rewrite rule $L \rightarrowtriangle R \in S$ and a string graph matching $m : L \to G$ such that

$$\begin{array}{ccccc} L & \xleftarrow{i} & I & \xrightarrow{j} & R \\ {\scriptstyle m}\downarrow & & \downarrow & & \downarrow \\ G & \hookleftarrow & D & \hookrightarrow & H \end{array}$$

is a string graph rewrite.

If we define

$$S = \{G \rightarrowtriangle_{i,j} H | G \approx_{i,j} H \in E\}$$

then $\rightarrowtriangle_S$ implements the Leibniz and Axiom rules of the previously-presented equational logic for string graphs. In particular, string graph rewriting implements a single application of Axiom followed by a single application of Leibniz (which, by proposition 3.5.2, is equivalent to any proof tree containing just Axiom and Leibniz axioms).

It then follows that $\overset{*}{\leftrightarrowtriangle}_S$, the reflexive, symmetric, transitive closure of $\rightarrowtriangle_S$, implements the previously presented equational logic with Homeo omitted.

3.6.2 Matching up to Wire Homeomorphism

There is one more rule we need to account for in our mechanisation process: HOMEO. Ideally, we would like to do rewriting "up to wire homeomorphism".

Given a string graph rewrite rule $L \rightarrowtriangle R$ and a graph G, we say that G rewrites to H by $L \rightarrowtriangle R$ *up to wire homeomorphism* if there are G', H' and $L' \rightarrowtriangle R'$, wire homeomorphic to G, H and $L \rightarrowtriangle R$ respectively, such that G' rewrites to H' by $L' \rightarrowtriangle R'$.

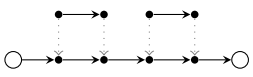
We would like to find the set $\mathcal{H}$ of graphs H such that G rewrites to H by $L \rightarrowtriangle R$ up to wire homeomorphism, quotiented out by wire homeomorphism (ie: so there are no two graphs in $\mathcal{H}$ wire homeomorphic to each other).

We can eliminate duplicate wire homeomorphic graphs as a last step by settling on some standard form (such as have at most one wire-vertex on each wire other than bare wires, where we must have two). However, this does not help to ensure we have found all possible matches without trying the (likely infinite number of) graphs wire homeomorphic to G. We demonstrate a technique for constraining the search space at the matching step by making a careful choice of G' and $L' \rightarrowtriangle R'$.

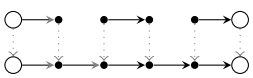
Any monomorphism from L to G must necessarily map wires of L onto wires of G of the same type in the following manner:

- A circle of L must map to a circle of G, in which case nothing else can map to that same circle
- An interior wire of L can only map to an interior wire of G, and this must be the only thing mapping to the wire.
- A bare wire of L can map to **any** wire of G, and may not be the only thing to map to that wire
- An input wire of L can map to either an input wire or an interior wire of G; in both cases, it may not be the only thing mapping to that wire
- Output wires are similar to input wires

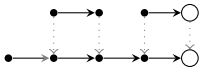
Note that multiple bare wires can match the same wire simultaneously



as well as matching an interior wire that has already been matched by an input and/or an output wire



or an input wire that has been matched by an input wire



or an output wire that has been matched by an output wire.

Let n be the number of bare wires in L. We choose L' to be the (unique, up to isomorphism) graph that is wire homeomorphic to L such that

- every circle has $\max(2n, 1)$ wire-vertices on it
- every interior wire has exactly $2n + 2$ wire-vertices on it
- every bare wire has no wire-vertices other than the input and output
- every input or output wire has no wire-vertices other than the input or output itself

We then choose G' to be the (unique, up to isomorphism) graph that is wire homeomorphic to G such that

- every circle has $\max(2n, 1)$ wire-vertices on it
- every interior wire has exactly $2n + 2$ wire-vertices on it
- every bare wire has exactly $2n$ wire-vertices on it (including boundary vertices)
- every input or output wire has exactly $2n + 1$ wire-vertices on it (including the boundary vertex)

We say that these graphs are *normalised*.

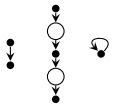
Any algorithm for finding graph monomorphisms (such as those in [40] or [2]) can then be used (together with a suitable filter) to find local isomorphisms from L' to $G' \setminus \mathrm{Bound}(G')$. Now consider an arbitrary wire w of G'.

- If w is a circle, it can be matched by any circle of the same type in L', since circles in both graphs have the same number of wire-vertices. Alternatively, if $n > 0$, it can be matched by any or all of the n bare wires of L'.
- If w is an interior wire, it can either be matched by one of the interior wires of L' (which have the same number of wire-vertices) or it can be matched by any combination of an input wire (consuming a single wire-vertex at one end), an output wire (similarly) and any or all of the bare wires (on the remaining $2n$ wire-vertices).
- If w is a bare wire, it can be matched by any or all of the n bare wires of L'.
- If w is an input wire, it can be matched by an input of L', which will consume one of the $2n + 1$ wire-vertices; this will still allow any or all of the bare wires of L' to match.
- Finally, if w is an output wire, it can similarly be matched by an output of L' and/or any or all of the bare wires of L'.

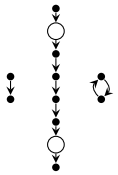
Rewriting using all of the matches found in this way will produce a finite collection $\mathcal{H}'$ of rewritten graphs. Contracting all the wires in these graphs so that there is only a single wire-vertex on each, except for bare wires which need two wire-vertices, and eliminating the duplicates will give us the set $\mathcal{H}$ we originally wanted.

Remark 3.6.11. The proliferation of wire-vertices and edges between them in L' and G' when L has bare wires means that this is not the most efficient way of doing matching up to wire homeomorphism. Quantomatic takes a slightly different approach, initially assuming $n = 0$ and delaying matching bare wires from L' until everything else has been matched; each bare wire is then matched against an edge of G', which is replaced by two wire-vertices and three edges at this point. See section 7.2 for details.

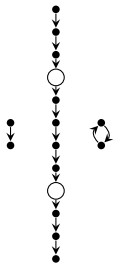
Example 3.6.12. The graph



would be normalised to



if it were the graph L from the rule and to



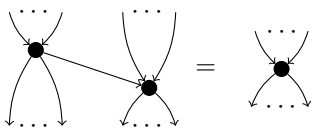
if it were the target graph G.

Chapter 4

!-Graphs

String graph rewrite systems have a shortcoming in that they do not have a finite way of representing infinitary rewrite rules. In this chapter, we present !-graphs (pronounced "bang graphs"), an extension of string graphs, as a way of expressing certain infinite families of string graphs in a finite way. In the next chapter, we will use these to construct infinite families of string graph rewrite rules and use them to rewrite string graphs, both directly and by rewriting !-graphs.

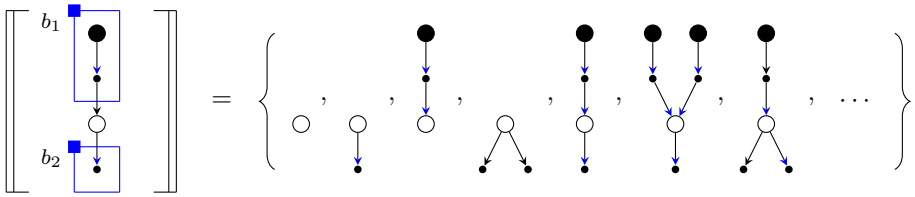
Recall the spider laws of section 2.2 that arose from commutative Frobenius algebras (CFAs):



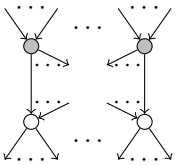
(4.1)

In a single diagram, this describes an infinite family of equations. However, while its meaning may be clear to a human, we need a more precise encoding for a proof assistant to be able to handle equations like this. In [9], Dixon and Duncan briefly described a system of *graph patterns* to handle laws like this. However, they provided no mathematical foundation for these. Kissinger, Merry and Soloviev provided such a foundation in [26] (as *pattern graphs*), making the system more powerful in the process. This joint work with the author of this thesis, which contained no proofs, forms the basis of this chapter, although we use the term !-*graphs* instead of pattern graphs to avoid confusion when describing matching (where the two graphs are often called the pattern and target graphs).

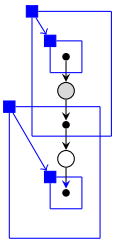
!-graphs formalise the notion of repetition indicated by the ellipses and encode it in the graph itself. What will actually be encoded is a collection of labelled subgraphs called !-boxes. A !-graph can be instantiated into a string graph by removing ("killing") or duplicating ("copying") these subgraphs (including incoming and outgoing edges) some (finite) number of times. In this way, a !-graph can stand in the place of an infinite family of string graphs:



We can even express the bipartite graph from the generalised bialgebra law



by nesting !-boxes inside other !-boxes:

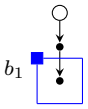


Section 4.4.1 goes into detail about the effect of nested and overlapping !-boxes.

4.1 Open Subgraphs

Note: when we refer to a subgraph of a string graph in this section, we mean a subgraph in $\mathbf{SGraph}_T$ – ie: the subgraph is also a string graph.

It should be clear that we cannot allow !-boxes to be arbitrary subgraphs. Copying b_1 in the following graph, for example, will lead to an invalid string graph, as there will be a wire vertex with multiple outputs:



Copying or killing a !-box must also not affect the fixed-arity edges of node-vertices, otherwise the arity-matching property of the typing morphism will be broken.

So we must restrict our !-boxes to those that can be copied. For string graphs, the correct notion is that of *open subgraphs*. Note that the definition here differs from, but is equivalent to, the one in [26] (pp 5).

Definition 4.1.1 (Open Subgraph)**.** A subgraph O of a string graph G is said to be *open* if it is not adjacent to any wire-vertices or incident to any fixed-arity edges.

This is also related to boundary-coherence in **SGraph**:

Proposition 4.1.2. *Let G be a string graph, and H a subgraph of G in $\mathbf{SGraph}_T$. Then H is open if and only if it has a complement $G \setminus H$ (in $\mathbf{SGraph}_T$), and the embedding of this complement into G is boundary-coherent.*

Proof. Suppose (H, τ_H) is open in (G, τ_G), and consider $(G \setminus H, \tau_{G \setminus H})$, which is a graph in $\mathbf{Graph}/\mathcal{G}_T$. Let n be a node-vertex in $G \setminus H$, and let $\tau^n_{G \setminus H}$ be the restriction of $\tau_{G \setminus H}$ to the fixed edge neighbourhood of n, and similarly for τ^n_G. Since $G \setminus H$ is a subgraph of G, $\tau_{G \setminus H}$ is a restriction of τ_G. So

$\tau^n_{G\setminus H}$ is a restriction of τ^n_G, and hence is injective and its image consists only of fixed-arity edges of the typegraph. Suppose it is not surjective, so there is some fixed-arity edge e incident to $\tau_{G\setminus H}(n)$ that is not in the image of $\tau_{G\setminus H}$. Then the preimage of e under τ_G, e', must not be in $G \setminus H$. Since n is in $G \setminus H$, and $G \setminus H$ is full in G, the other end of e', which we will call w, must be in H. But then H is incident to a fixed-arity edge (namely e'), and hence is not open in G. So $G \setminus H$ is a string graph, since (being a subgraph of G) it must have at most one incoming and one outgoing edge on each wire-vertex.

For boundary-coherence of the embedding $\iota : G \setminus H \to G$, we need to show that any input of $G \setminus H$ is also an input of G, and likewise for outputs. But the only way an input w in $G \setminus H$ could not be an input in G is if there is an edge e in G from some vertex v to w. Because e is not in $G \setminus H$, neither can v be. So v must be in H, and so H is incident to a wire-vertex (w), contradicting the openness of H. The argument for outputs is analagous, and so we have boundary-coherence as required.

Conversely, suppose (H, τ_H) is a (string) subgraph of (G, τ_G), and that its complement is also a string graph with a boundary-coherent embedding ι into G. Suppose H is adjacent to a wire-vertex w, via an edge e. Then w is in $G \setminus H$, but e is not. If w is the target of e, w must be an input of $G \setminus H$ but not of G. Otherwise, w is the source of e and w is an output of $G \setminus H$ but not of G. Either way, ι cannot be boundary-coherent. So there is no such w.

Now suppose H is incident to a fixed-arity edge e. Let v be the end of e that is not in H. v cannot be a wire-vertex, since we have already shown that H cannot be adjacent to a wire-vertex. So v is a node-vertex. Let $e' = \tau_G(e)$. Then e' is in the fixed neighbourhood of $\tau_G(v)$, but is not in the image of $\tau_{G\setminus H}$. So $\tau_{G\setminus H}$ is not arity-matching, contradicting the assumption that $(G \setminus H, \tau_{G\setminus H})$ is a string graph. Thus there can be no such edge e, and H is open in G. □

The use of the term "open" brings with it the expectation that the property is preserved under unions and intersections (since we are dealing with finite graphs). The following proposition shows that this is, indeed, the case.

Proposition 4.1.3 ([26], pp 5)**.** *If $O, O' \subseteq G$ are open subgraphs, and $H \subseteq G$ is an arbitrary subgraph, then $O \cap O'$ and $O \cup O'$ are open in G and $H \cap O$ is open in H.*

Proof. Proposition 3.3.10 gives us that the graphs are all string graphs.

Suppose $O \cap O'$ is adjacent to a wire-vertex. Then there is a wire-vertex w in G but not in $O \cap O'$ adjacent to a vertex v in $O \cap O'$. If w is not in O, we have a contradition for the openness of O. So w must be in O. Similarly, w must be in O'. But then w is in $O \cap O'$, which contradicts our assumption. So there can be no such w. Instead, suppose $O \cap O'$ is incident to a fixed-arity edge e. Again, e must be in O, since O cannot be incident to a fixed-arity edge, and must also be in O' for

the same reason, and hence is in $O \cap O'$, and so not incident to it. So there is no such e and $O \cap O'$ is open in G.

Suppose $O \cup O'$ is adjacent to a wire-vertex. Then there is a wire-vertex w in G but not in $O \cup O'$ adjacent to a vertex v in $O \cup O'$. v must either be in O or O'; WLOG assume it is in O. Then, since w is not in O, O is adjacent to a wire-vertex. But this cannot be the case, since O is open, so there can be no such w. Similarly, if there is a fixed-arity edge e incident to $O \cup O'$, it must be incident to either O or O', which cannot be the case. So there is no such e, and $O \cup O'$ is open in G.

Suppose there is a wire-vertex w in H but not in $H \cap O$ adjacent to a vertex v in $H \cap O$. Then w is in G but not in O, and v is in O, and they are joined by an edge in G. So O is adjacent to a wire-vertex in G, contradicting its openness in G, so there is no such w. Suppose now that $H \cap O$ is incident to a fixed-arity edge e in H, and let v be the vertex in $H \cap O$ it is incident to. Then v is in O, and e is in G but not O, and so O cannot be open in G, which contradicts the assumption. So there can be no such e, and $H \cap O$ is open in H.

□

4.2 The Category of !-Graphs

The !-boxes are encoded into the graph itself as a new vertex type. These !-*vertices* act as the labels for the !-boxes, and the edges from the !-vertices to other graph vertices indicate the !-box contents. We therefore need to extend the derived typegraph of a monoidal signature T to include this new vertex.

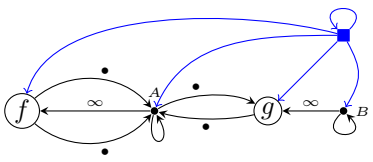
Definition 4.2.1 (Derived Compressed !-typegraph)**.** The *derived compressed* !*-typegraph* $\mathcal{G}_{T!}$ of a compressed monoidal signature T has the same vertices as $\mathcal{G}_T$ (see definition 3.3.5) together with a distinct vertex !, and the same edges as $\mathcal{G}_T$ together with a self-loop $p_!$ on ! and, for each vertex v of $\mathcal{G}_T$, an edge p_v from ! to v.

Note that the definition of $\mathcal{G}_{T!}$ ensures that there can be no edges from wire- or node-vertices to !-vertices.

The derived compressed !-typegraph of the compressed monoidal signature

$$f : [A^\infty] \to [A^\bullet, A^\bullet]$$
$$g : [B^\infty, A^\bullet] \to [A^\bullet]$$

would be



The wire-vertex and node-vertex terminology is inherited from string graphs. In addition, for a graph $(G, \tau : G \to \mathcal{G}_{T!})$ in $\mathbf{Graph}/\mathcal{G}_{T!}$, we call any vertex v where $\tau(v) = !$ a !-*vertex*, and denote the set of all !-vertices in G as $!(G)$.

We alter the definition of an *input* slightly from the string-graph case, due to the new vertex type: a wire-vertex is an input if the only in-edges are from !-vertices.

For a !-vertex $b \in !(G)$, let $B(b)$ be its associated !-box. This is the full subgraph whose vertices are the set $\mathrm{succ}(b)$ of all of the successors of b. We also define the parent graph of a !-vertex $B^{\uparrow}(b)$ as the full subgraph of predecessors, that is, the full subgraph generated by $\mathrm{pred}(b)$. Note that the typegraph constrains $B^{\uparrow}(b)$ to only contain !-vertices (and edges between them).

As a notational convenience, when dealing with subgraphs of $\mathcal{G}_{T!}$-typed graphs where the inclusion map is implicit, we will use subscripts to identify which graph we are referring to. So if H is a subgraph of G and b is a !-vertex in H, $B_H(b)$ will be its !-box in H, and $B_G(b)$ will be the !-box of b when considered as a vertex of G.

Note that, for a given monoidal signature T, $\mathcal{G}_T$ is a subgraph of $\mathcal{G}_{T!}$. Since **Graph** has pullbacks, theorem 3.3.2 gives us a full, coreflective embedding $E : \mathbf{Graph}/\mathcal{G}_T \to \mathbf{Graph}/\mathcal{G}_{T!}$, whose right adjoint is the forgetful functor $U : \mathbf{Graph}/\mathcal{G}_{T!} \to \mathbf{Graph}/\mathcal{G}_T$ that simply drops all !-vertices.

Definition 4.2.2 (!-Graph and $\mathbf{BGraph}_T$; [26], pp 5)**.** A $\mathcal{G}_{T!}$-typed graph G is called a !-*graph* if:

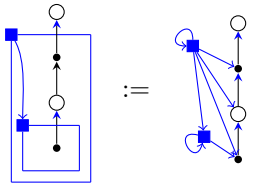
1. $U(G)$ is a string graph;
2. the full subgraph with vertices $!(G)$, denoted $\beta(G)$, is posetal;
3. for all $b \in !(G)$, $U(B(b))$ is an open subgraph of $U(G)$; and
4. for all $b, b' \in !(G)$, if $b' \in B(b)$ then $B(b') \subseteq B(b)$.

$\mathbf{BGraph}_T$ is the full subcategory of $\mathbf{Graph}/\mathcal{G}_{T!}$ whose objects are !-graphs.

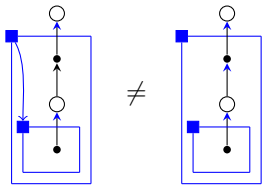
Recall that a graph is posetal if it is simple (at most one edge between any two vertices) and, when considered as a relation, forms a partial order. Note in particular that this implies $b \in B(b)$ (and $B^{\uparrow}(b)$), by reflexivity. This partial order allows !-boxes to be nested inside each other, provided that the subgraph defined by a nested !-vertex is totally contained in the subgraph defined by its parent (condition 4).

As with string graphs, we only consider finite !-graphs in the sequel.

We introduce special notation for !-graphs. !-vertices are drawn as squares, but rather than drawing edges to all of the node-vertices and wire-vertices in $B(b)$, we simply draw a box around it.



In this notation, we retain edges between distinct !-vertices to indicate which !-boxes are nested as opposed to simply overlapping. This distinction is important, as nested !-boxes are copied whenever their parent is copied.



We will prove a few useful results about !-graphs and about $\mathbf{BGraph}_T$. Firstly, we have a sufficient condition for a subgraph of a !-graph to be a !-graph.

Lemma 4.2.3. *Let G be a !-graph and H a full subgraph of G such that $U(H)$ is a string graph. Then H is a !-graph.*

Proof. $U(H)$ is a string graph by assumption.

Since H is full in G, $\beta(H)$ must be a full subgraph of $\beta(G)$. But a full subgraph of a posetal graph is itself posetal, so $\beta(H)$ is posetal.

Let $b \in !(H)$. Then $U(B_G(b))$ is an open subgraph of $U(G)$, and $U(B_H(b)) = U(B_G(b)) \cap U(H)$, which is open in H by proposition 4.1.3.

Let $b, c \in !(H)$, with $c \in B_H(b)$. Then we must have $c \in B_G(b)$, since G is a !-graph. So $B_G(c) \subseteq B_G(b)$. Since H is full in G, we know that $B_H(b) = B_G(b) \cap H$ and similarly for c. Then $B_G(c) \cap H \subseteq B_G(b) \cap H$, so $B_H(c) \subseteq B_H(b)$.

So H is a !-graph, as required. □

Corollary 4.2.4. *Let G be a !-graph, and H a subgraph of G such that $U(H)$ is open in $U(G)$. Then $G \setminus H$ is a !-graph.*

It is also useful to know under what conditions a !-graph morphism is monic or (strongly) epic in $\mathbf{BGraph}_T$. These conditions turn out to be the same as for $\mathbf{SGraph}_T$.

Lemma 4.2.5. *A morphism in $\mathbf{BGraph}_T$ is monic iff it is injective.*

Proof. Since this holds in $\mathbf{Graph}/\mathcal{G}_{T!}$, any injective map in $\mathbf{BGraph}_T$ must be monic in $\mathbf{Graph}/\mathcal{G}_{T!}$, and hence also monic in $\mathbf{BGraph}_T$.

Suppose we have a non-injective morphism $f : G \to H$ in $\mathbf{BGraph}_T$. Then f must either map two or more edges in G to the same edge in H or map two or more vertices in G to the same vertex in H. In fact, since G and H are both simple, f must map two or more vertices v_i in G to a single vertex v in H. Consider the smallest sub-!-graph K of H containing $f(v_i)$. This is the vertex itself, plus a self-loop if it is a !-vertex or any fixed-arity edges (plus their terminating wire-vertices) if it is a node-vertex. Then for each v_i, we construct the !-graph morphism g_i that takes the single vertex in K to v_i (this will extend to any incident edges of the vertex in K in a unique way, dictated by the

typing morphisms). Now all the $f \circ g_i$ are the same morphism, but the g_i morphisms are distinct, so f is not monic. □

Lemma 4.2.6. *Let $f : G \to H$ be a !-graph morphism. Then $f[G]$, the image of f, is a !-graph.*

Proof. First we note that $f[G]$ is a subgraph of H, and let $\iota : f[G] \to H$ be the inclusion. Then $U(f[G])$ is a subgraph of the string graph $U(H)$. It therefore satisfies the wire-vertex conditions for string graphs, and we only need check that τ_H restricts to an arity-matching morphism. But we know that f is arity-matching, and hence restricts to a bijection on the fixed-arity neighbourhood of every node-vertex v in G and its image in H, and τ_G is also arity-matching, and τ_G and $\tau_H \circ f$ commute. So $\tau_H \restriction_{f[G]}$ must be arity-matching, and hence $U(f[G])$ must be a string graph.

$\beta(f[G])$ is a subgraph of $\beta(H)$. But then $\beta(f[G])$ is simple and anti-symmetric, since H is, and is reflexive and transitive, since G is, and hence is posetal.

Let $b' \in !(f[G])$. Then there is a $b \in !(G)$ with $\iota(b') = f(b)$. Now $B(b') = B(f(b)) \cap f[G]$, which is open in $f[G]$ by proposition 4.1.3.

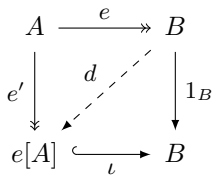
Let $b, c \in !(f[G])$, with $c \in B(b)$. Then we must have $\iota(c) \in B(\iota(b))$. So $B(\iota(c)) \subseteq B(\iota(b))$. But then $B(b) = B(\iota(b)) \cap f[G]$ and similarly for c, and $B(\iota(c)) \cap f[G] \subseteq B(\iota(b)) \cap f[G]$, so $B(c) \subseteq B(b)$.

So $f[G]$ is a !-graph, as required. □

Lemma 4.2.7. *A morphism in $\mathbf{BGraph}_T$ is a strong epimorphism iff it is surjective.*

Proof. Since this holds in $\mathbf{Graph}/\mathcal{G}_{T!}$, any surjective map in $\mathbf{BGraph}_T$ must be a strong epimorphism in $\mathbf{Graph}/\mathcal{G}_{T!}$, and hence a strong epimorphism in $\mathbf{BGraph}_T$.

Now suppose $e : A \to B$ is a strong epimorphism. Let $e' : A \to e[A]$ be the restriction of e to its image (which is also a !-graph by lemma 4.2.6) and let $\iota : e[A] \to B$ be the inclusion of the image of e in B. Then ι is monic, so there exists a map d making the following diagram commute:



But then $\iota \circ d = 1_B$, so ι must be surjective and hence so is e. □

Finally, it is worth noting how $\mathbf{SGraph}_T$ and $\mathbf{BGraph}_T$ are related.

Definition 4.2.8 (Concrete Graph)**.** A !-graph with no !-vertices is called a *concrete graph.*

The full subcategory of $\mathbf{BGraph}_T$ consisting of concrete graphs is, in fact, the image of $\mathbf{SGraph}_T$ under the previously-mentioned embedding functor E. Concrete graphs and string graphs will therefore be considered interchangable.

Indeed, the restriction of E to $\mathbf{SGraph}_T$ is a full, coreflective embedding, and its right adjoint is the restriction of U to $\mathbf{BGraph}_T$. As a result, we will also use E and U to refer to these restrictions; which is meant should be clear from context. We use Σ to mean $E \circ U$ (and note that $\mathbf{BGraph}_T$ is closed under Σ).

We can extend the $\beta(G)$ notation of definition 4.2.2 to morphisms of $\mathbf{Graph}/\mathcal{G}_{T!}$ (and hence of $\mathbf{BGraph}_T$) by making their operation be the obvious restrictions. More precisely, if $\mathcal{G}_!$ is the subgraph of $\mathcal{G}_{T!}$ consisting of only the !-vertex and its self-loop (with ι_T being its inclusion map), we can view β as $E_{\iota_T} \circ U_{\iota_T}$ (in the terminology of theorem 3.3.2).

4.2.1 Wire Homeomorphisms

We will need to extend our notion of wire-homeomorphism to !-graphs.

A *wire homeomorphism* $f : G \sim H$ between two !-graphs G and H consists of four bijective type-preserving functions

- $f_N : N(G) \leftrightarrow N(H)$
- $f_! : !(G) \leftrightarrow !(H)$
- $f_B : \text{Bound}(G) \leftrightarrow \text{Bound}(H)$
- $f_W : \text{Wires}(G) \leftrightarrow \text{Wires}(H)$

such that

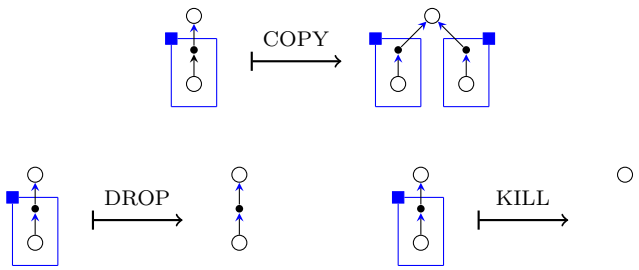
- for each wire w in $\text{Wires}(G)$ with a source v in $N(G)$ (resp. $\text{Bound}(G)$), the source of $f_W(w)$ in H is $f_N(v)$ (resp. $f_B(v)$), and similarly for targets of wires
- for each $b \in !(G)$, there is an edge in G from b to a vertex v in $!(G)$ (resp. $N(G)$, $\text{Bound}(G)$) if and only if there is an edge in H from $f_!(b)$ to $f_!(v)$ (resp. $f_N(v)$, $f_B(v)$)
- for each $b \in !(G)$ and each wire $w \in \text{Wires}(G)$, there is an edge in G from b to each $v \in w_V$ if and only if there is an edge in H from $f_!(b)$ to each $v \in (f_W(w))_V$

This is the same as the definition for string graphs, but with a map for !-boxes and some constraints to preserve !-box containment. This means that if $f : G \sim H$ is a wire-homeomorphism of !-graphs, discarding $f_!$ will give a wire-homeomorphism $U(G) \sim U(H)$.

4.3 The !-Box Operations

We have already mentioned that !-boxes are intended to be subgraphs that can be copied or deleted. The precise set of operations allowed on a !-box are the following ([26], pp 6, although we omit

MERGE from that list and revisit it in section 6.5):



These are inspired by the rules for the "bang" operation from linear logic[16]. In particular, COPY corresponds to contraction, DROP to dereliction and KILL to weakening.

Definitions 4.3.1 (!-Box Operations; [26], pp 6)**.** For G a !-graph and $b \in !(G)$, the three !-box operations are defined as follows:

- $\mathrm{COPY}_b(G)$ is defined by a pushout of inclusions in $\mathbf{Graph}/\mathcal{G}_{T^!}$:

$$\begin{array}{ccc} G \setminus B(b) & \hookrightarrow & G \\ \downarrow & & \downarrow \\ G & \longrightarrow & \mathrm{COPY}_b(G) \end{array} \tag{4.2}$$

- $\mathrm{DROP}_b(G) := G \setminus b$
- $\mathrm{KILL}_b(G) := G \setminus B(b)$

Before we show that these operations preserve the property of being a !-graph, it will be useful to adapt the notion of boundary-coherence to !-graphs. The important thing is that, in addition to the restrictions on how the morphisms interact with the boundaries of the graphs, we also need to restrict how they interact with !-box containment. Note that the definition of boundary-!-coherence we give here is stronger than is actually needed for producing pushouts of !-graphs; this gives us a simpler definition and simpler proofs.

Definition 4.3.2. Let $f : G \to H$ be a !-graph morphism. f is said to *reflect* !*-box containment* if whenever we have an edge e in H whose source is a !-vertex and whose target is in the image of f, e is also in the image of f.

A span of monomorphisms $G \xleftarrow{g} I \xrightarrow{h} H$ in $\mathbf{BGraph}_T$ is called *boundary-!-coherent* if the span formed from $U(g)$ and $U(h)$ is boundary-coherent and both g and h reflect !-box containment.

As in $\mathbf{SGraph}_T$, a single morphism $f : G \to H$ of string graphs is called boundary-!-coherent if the span $H \xleftarrow{f} G \xrightarrow{f} H$ is.

The main purpose of this definition is to provide us with a sufficient condition for a span to have a pushout in $\mathbf{BGraph}_T$.

Proposition 4.3.3. *If a span of monomorphisms $G \xleftarrow{g} I \xrightarrow{h} H$ in* $\mathbf{BGraph}_T$ *is boundary-!-coherent, then it has a pushout in* $\mathbf{BGraph}_T$.

Proof. Suppose g and h are boundary-!-coherent. Since they are monomorphisms, they have a pushout in $\mathbf{Graph}/\mathcal{G}_{T!}$:

$$\begin{array}{ccc} I & \xrightarrow{h} & H \\ {\scriptstyle g}\downarrow & & \downarrow{\scriptstyle p_h} \\ G & \xrightarrow[p_g]{} & D \end{array} \tag{4.3}$$

It suffices to show that D is a !-graph.

The span formed by $U(g)$ and $U(h)$ is boundary-coherent, so the graph that results from pushing out this span in $\mathbf{Graph}/\mathcal{G}_T$ is in $\mathbf{SGraph}_T$ (theorem 3.6.7), and so is a string graph. But this is just $U(D)$, by proposition 3.3.4.

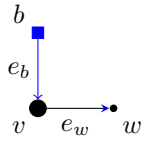
Since all the morphisms are monic (and the square commutes), we can treat them as containment relations. So we will consider G and H to be subgraphs of D, and I to be their common subgraph.

Now we note that if we have an edge e_b from a !-vertex b to a vertex v in D, and another edge e incident to v in D, both edges must be in G or else both edges must be in H. To see this, suppose not. Then one edge (e_b, say) must be in G and the other (e) in H. So v must be in I, since it is in both G and H. But then, as g reflects !-box containment, e_b must be in I and hence in H.

We need to show that $\beta(D)$ is posetal. $\beta(D)$ must be simple, since if we have two edges from a !-vertex b_1 to a !-vertex b_2 in D, they must both be in G or both in H by the above reasoning, which cannot be the case as these are !-graphs. So there are no such edges. Anti-symmetry is shown in the same way by reversing one of the edges. Reflexivity is inherited from G and H. Transitivity follows by considering that if we have the following pair of edges

they must both be in G or both in H, and hence there must be an edge from b_1 to b_3 inherited from that graph. So $\beta(D)$ is posetal.

Let $b \in !(D)$, and suppose $U(B(b))$ is not open in $U(D)$. Then there is either a wire-vertex adjacent to $U(B(b))$, or a fixed-arity edge incident to it. The former case is impossible, because if we have two edges such that



where w is a wire-vertex, then e_w and e_b must both be in G or both be in H, and hence there must be an edge from b to w. Similarly, if there is a fixed-arity edge e incident to v, e_b and e must be both in G or both in H, and hence there must be an edge from b to the other end of e, and so e is not incident to $U(B(b))$.

Let $b, c \in !(D)$ with $c \in B(b)$, and let $v \in B(c)$. We need to show that $v \in B(b)$. But the edge from b to c and the edge from c to v must both be in G or both be in H, and so there must be an edge from b to v, as required. □

We now proceed to prove that all the !-box operations produce !-graphs. This theorem was stated, but not proved, in [26] (pp 7).

Theorem 4.3.4. *Let G be a !-graph and $b \in !(G)$. Then the $\mathcal{G}_{T!}$-typed graphs $\mathrm{COPY}_b(G)$, $\mathrm{DROP}_b(G)$ and $\mathrm{KILL}_b(G)$ are all !-graphs.*

Proof. Corollary 4.2.4 gives us that $G \setminus B(b)$ is a !-graph, and hence its embedding ι into G is a !-graph monomorphism. This embedding is boundary-!-coherent: since $U(B(b))$ is an open subgraph of $U(G)$, proposition 4.1.2 gives us that $U(\iota)$ is a boundary-coherent morphism, and requirement 4 of definition 4.2.2 ensures that ι reflects !-box containment. So we know the pushout exists in $\mathbf{BGraph}_T$ by proposition 4.3.3, and hence $\mathrm{COPY}_b(G)$ is a !-graph.

The DROP and KILL cases follow from lemma 4.2.3 and corollary 4.2.4 respectively. □

Since one of the reaons for using string graphs is their ability to model inputs and outputs, it is useful to know how these operations affect the inputs and outputs of !-graphs. Firstly, we extend $\mathrm{In}(G)$, $\mathrm{Out}(G)$ and $\mathrm{Bound}(G)$ to include !-boxes in the following way:

Definition 4.3.5. For a !-graph G, we define $\mathrm{In}_!(G)$ to be the full subgraph of G containing the vertices $!(G)$ and $\mathrm{In}(U(G))$; $\mathrm{Out}_!(G)$ to be the full subgraph containing $!(G)$ and $\mathrm{Out}(U(G))$; and $\mathrm{Bound}_!(G)$ is the subgraph containing $!(G)$ and $\mathrm{Bound}(U(G))$, and any edges with a source in $!(G)$ and a target in either $!(G)$ or $\mathrm{Bound}(U(G))$.

These are all valid !-graphs. Now we can show that the !-graph operations affect the boundary in the expected manner.

Lemma 4.3.6. *Let G be a !-graph, and $b \in !(G)$, and consider the copy operation:*

$$\begin{array}{ccc} G \setminus B(b) & \xrightarrow{i_1} & G \\ {\scriptstyle i_2}\downarrow & & \downarrow{\scriptstyle p_1} \\ G & \xrightarrow[p_2]{} & \mathrm{COPY}_b(G) \end{array}$$

Then all the maps in the pushout take inputs to inputs and outputs to outputs.

Proof. The square is symmetric, and the cases for inputs mirrors the case for outputs, so we just consider inputs for i_1 and p_1.

That i_1 maps inputs to inputs follows from the fact that it is boundary-!-coherent.

Suppose $w \in \mathrm{In}(G)$, and consider $p_1(w)$. If there is an edge e in $\mathrm{COPY}_b(G)$ with $t(e) = p_1(w)$, then e cannot be in the image of p_1, since w is an input of G and p_1 is injective. So e, and hence

$p_1(w)$, must be in the image of p_2. But then there must be a vertex w' in $G \setminus B(b)$ with $i_1(w') = w$ and $p_2(i_2(w')) = p_1(w)$, since this is a pushout. w' must be an input of $G \setminus B(b)$, since if there were an incoming edge to w', it would have to map to an incoming edge of w in G, and there is no such edge. So $i_2(w')$ must also be an input of G, since i_2 maps inputs to inputs. Then $i_2(w')$ cannot have an incoming edge, and so e cannot be in the image of p_2. So there is no such edge e, and $p_1(w) \in \mathrm{In}(\mathrm{COPY}_b(G))$. □

Theorem 4.3.7. *Let G be a !-graph, and $b \in !(G)$. Then*

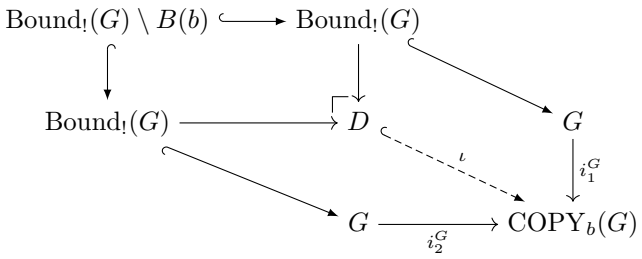
- $\mathrm{Bound}_!(\mathrm{COPY}_b(G)) \cong \mathrm{COPY}_b(\mathrm{Bound}_!(G))$*;*
- $\mathrm{Bound}_!(\mathrm{KILL}_b(G)) \cong \mathrm{KILL}_b(\mathrm{Bound}_!(G))$*; and*
- $\mathrm{Bound}_!(\mathrm{DROP}_b(G)) \cong \mathrm{DROP}_b(\mathrm{Bound}_!(G))$

and similarly for $\mathrm{In}_!$ *and* $\mathrm{Out}_!$*.*

Proof. For the COPY_b case, since $\mathrm{Bound}_!(G)$ is a !-graph, we can construct the pushout

$$\begin{array}{ccc} \mathrm{Bound}_!(G) \setminus B(b) & \hookrightarrow & \mathrm{Bound}_!(G) \\ \downarrow & & \downarrow \\ \mathrm{Bound}_!(G) & \longrightarrow & D \end{array}$$

in $\mathbf{BGraph}_T$, where $D = \mathrm{COPY}_b(\mathrm{Bound}_!(G))$. This is a subgraph of $\mathrm{COPY}_b(G)$:



where i_1^G and i_2^G are the inclusions from the pushout that defines $\mathrm{COPY}_b(G)$.

Now every edge and vertex in $\beta(\mathrm{COPY}_b(G))$ is in the image of either i_1^G or i_2^G, and its preimage is in $\mathrm{Bound}_!(G)$. Then it must be in the image of ι, so D contains all !-vertices and edges between them from $\mathrm{COPY}_b(G)$.

Now let w be a boundary vertex of $\mathrm{COPY}_b(G)$. It must have a preimage under at least one of i_1^G and i_2^G, and lemma 3.3.9 gives us that any such preimage in G is also a boundary vertex, and hence in $\mathrm{Bound}_!(G)$, and so in the image of ι. Conversely, if w is a wire-vertex in the image of ι, it must be in the image of a boundary vertex of G under either i_1^G or i_2^G. Then lemma 4.3.6 gives us that w is a boundary-vertex of $\mathrm{COPY}_b(G)$.

If e is an edge from a !-vertex to a boundary vertex in $\mathrm{COPY}_b(G)$, it must be in the image of either i_1^G or i_2^G, and its preimage e' under these map(s) must likewise be an edge from a !-vertex

to a boundary vertex in G. So e' is in $\text{Bound}_!(G)$, and hence in the image of ι, and we have that $\iota[D] = \text{Bound}_!(\text{COPY}_b(G))$.

For the remaining cases, we note that $\beta(\text{Bound}_!(G)) = \beta(G)$. It then follows that

$$\beta(\text{Bound}_!(\text{KILL}_b(G))) = \beta(\text{KILL}_b(\text{Bound}_!(G)))$$

and similarly for DROP_b. Then the result for KILL_b follows from the fact that $U(B(b))$ is an open subgraph of $U(G)$, and DROP from the fact that it does not affect $U(G)$.

The arguments for $\text{In}_!$ and $\text{Out}_!$ are almost identical. □

4.4 Matching String Graphs With !-Graphs

A !-graph can be considered to be a pattern for string graphs, similar in spirit to a regular expression, although less powerful. Implicit in the term "pattern", though, is that there should be a notion of matching. We already have a notion of what it means for a string graph to match another string graph: the first string graph must be a subgraph of the second (with some additional constraints). So what does it mean for a !-graph to match a string graph?

If G is a !-graph, an obvious candidate for a string graph that should be matched by G is $U(G)$, the concrete part of the graph. Recalling the discussion at the start of the chapter, our aim is to match the concrete part of !-graphs with "any number of copies" of each !-box in G.

This is where the !-box operations come in. COPY_b and KILL_b provide a way of getting "any number of copies" (including zero) of the !-box $B(b)$. DROP_b then provides a way of producing a concrete graph (which we have already noted is essentially the same as a string graph). These provide the a notion of *instantiation*, by which we can produce a string graph from a !-graph.

Note that this definition is slightly different to the one in [26] (pp 7). The notions of *instance* and *instantiation* in that paper correspond to *concrete instance* and *concrete instantiation* here.

Definition 4.4.1 (Instantiation)**.** For !-graphs G, H, we let $G \succeq H$, and say H is an *instance* of G, if and only if H can be obtained from G (up to isomorphism) by applying the operations from definition 4.3.1 zero or more times. This sequence of operations is called an *instantiation* of H from G. If H is a concrete graph, it is called a *concrete instance* of G, and any instantiation of it is also called *concrete*.

Remark 4.4.2. Given an instantiation S of H from G, we will sometimes use the notation $S(G)$ to refer to H. Note that S is not really a function: because the first operation of S refers to a !-vertex of G, S cannot be applied to anything other than G (although we can sometimes relax this constraint in the presence of certain morphisms, as we will see in later chapters).

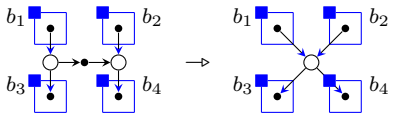
We can then say that a !-graph G matches a string graph H if and only if there is a concrete instance of G that matches H:

Definition 4.4.3 (Matching; [26], pp 8)**.** Let P be a !-graph, and H a string graph. If there is a concrete instance G of P, with instantiation S, and a monic local isomorphism $m : U(G) \to H$, P is said to *match* H *at* m *under* S, and m is said to be a *matching of* P *onto* H *under* S.

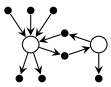
Note that all the !-box operations yield a pattern that is at least as specific; if we apply a !-box operation to a !-graph G to get another !-graph H, any string graph matched by H will also be matched by G. $\succeq$ can therefore be viewed as a refinement (pre-)ordering on !-graphs.

Given a !-graph P that satisfies a couple of constraints on !-boxes that ensure there are only ever a finite number of matchings onto any given graph (for example, there can be no empty !-boxes), it is possible to decide for any string graph G whether there is a matching of P onto G. An algorithm for this (that also produces the required instantiation) is given in chapter 7.

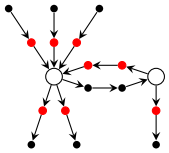
Example 4.4.4. Suppose we wish to use a rewrite rule implementing the Z spider law



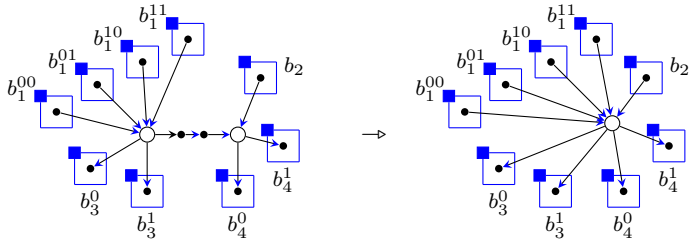
to rewrite



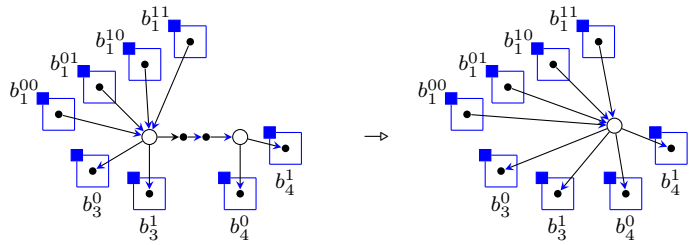
We start by normalising the target graph:



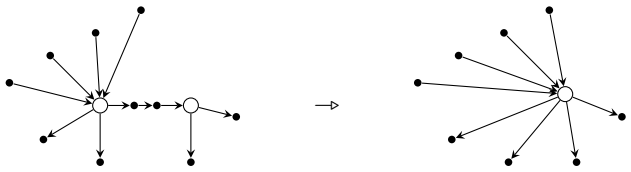
The highlighted wire-vertices are the ones that need to be matched by vertices that are in !-boxes in the LHS of the rule. So we create four copies of b_1 and two each of b_3 and b_4



then kill b_2



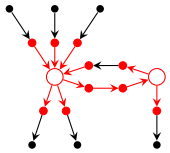
and drop all the remaining !-boxes



This corresponds to the instantiation

$$\mathrm{COPY}_{b_1}; \mathrm{COPY}_{b_1^0}; \mathrm{COPY}_{b_1^1}; \mathrm{COPY}_{b_3}; \mathrm{COPY}_{b_4}; \mathrm{KILL}_{b_2}; \mathrm{DROP}_{b_1^{00}};$$
$$\mathrm{DROP}_{b_1^{01}}; \mathrm{DROP}_{b_1^{10}}; \mathrm{DROP}_{b_1^{11}}; \mathrm{DROP}_{b_3^0}; \mathrm{DROP}_{b_3^1}; \mathrm{DROP}_{b_4^0}; \mathrm{DROP}_{b_4^1}$$

where, as a notational convenience, we refer to the copies of b in $\mathrm{COPY}_b(G)$ as b^0 and b^1 (in abitrary order). The resulting rule matches most of the graph:



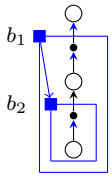
and rewriting (and removing extraneous wire-vertices) results in

4.4.1 Nested and Overlapping !-boxes

The effect of edges between !-vertices on the !-box operations deserves special attention. These edges can be thought of as indicating a parent-child relation, with the edge going from the parent !-box to the child !-box (although this view does not entirely hold up when we consider the self-loop that exists on each !-vertex).

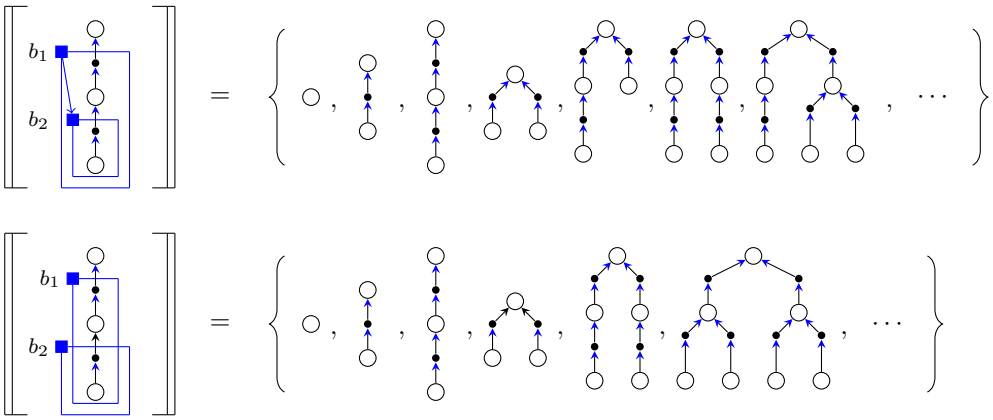
The edges from a !-vertex determine which other vertices should be copied or killed when the !-vertex has COPY or KILL, respectively, applied to it. So an edge from a !-vertex b to another !-vertex c means that COPY_b will also copy c (and its associated !-box).

An example that demonstrates the difference that an edge between !-vertices makes is the following:

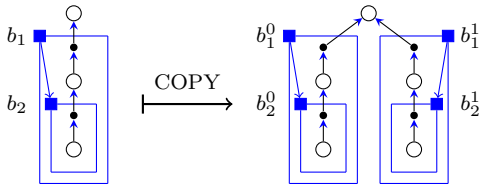


This will match any tree of white nodes of depth at most two. However, if we remove the edge

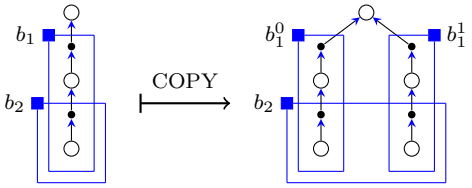
between the !-vertices, we get only the *balanced* trees.



The reason for this is that when we copy the outer !-box in the case where there is an edge, we also copy the inner !-box.

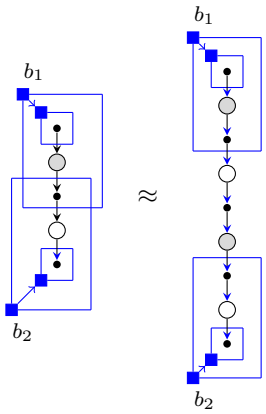


This produces a separate !-box round each second-level node, allowing different numbers of nodes on different branches (eg: one could copy b_2^0 and kill b_2^1). In the case without an edge, however, copying b_1 will simply extend b_2, and every copy of b_2 will have exactly one second-level node in each first-level branch of the tree.

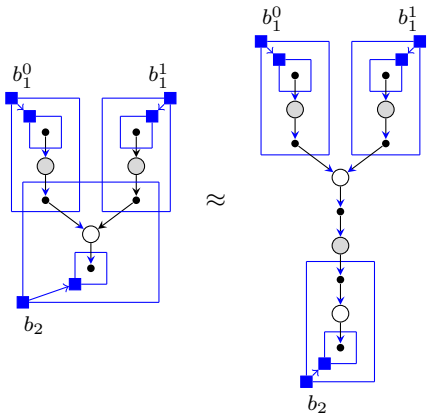


In this case, copying or killing b_2 would have the same effect on every branch of the tree, ensuring that it remains balanced.

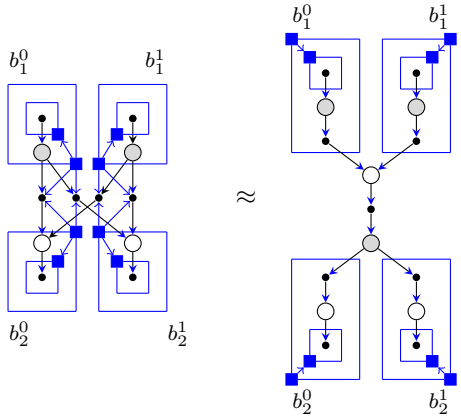
Example 4.4.5. Recall the generalised bialgebra graph equation from the start of the chapter:



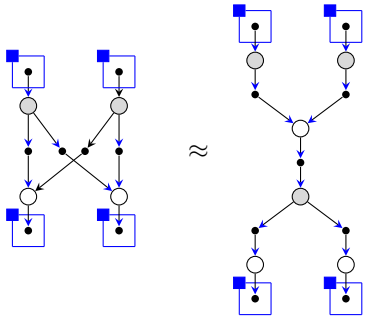
Ignoring, for the moment, the details of the structure used to represent this equation (which we detail in section 5.2) we can (informally) show that the spiderised bialgebra law is an instance of this. First, we apply COPY_{b_1}:



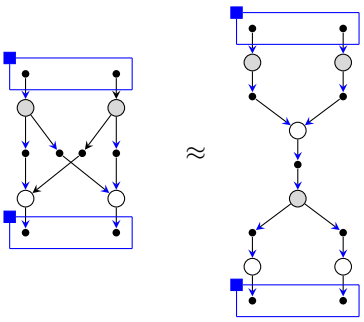
Next we apply COPY_{b_2}. At this point, our shorthand !-box notation causes more confusion than it alleviates, so we use a hybrid notation where we show some of the !-box containment edges explicitly.



Finally, we DROP all the top-level !-vertices, giving us the spiderised bialgebra law.



If the !-boxes had not been nested, we would have ended up with the weaker statement

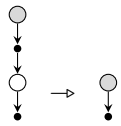


4.5 Related Work

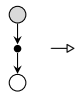
Boneva et al describe[4] a system of *graph shapes* to generalise over classes of graphs in the context of model checking. Their approach is similar to the typegraph constructions we introduced for string graphs; a shape is a graph with some multiplicity constraints on edges and vertices, and a graph G is a *concretisation* of a shape S if there is a morphism from G to S satisfying those constraints (for example, a node of S could require that exactly 3 nodes of G map to it, or an edge could require more than 5 edges to match it). There is a shape-wide degree of approximation in these multiplicities.

The aim of graph shapes is different from that of !-graphs, even though both attempt to capture families of graphs in a single graph, and this leads to some substantial differences between them. For example, graph shapes are mostly local (although there is some weak grouping of nodes); multiplicities, in particular, are specified on a per-edge or per-node basis. !-graphs, by contrast, operate on arbitrarily-large subgraphs.

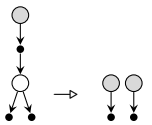
Taentzer introduced the idea of amalgamated graph transformations in [39]. These provide a framework for applying multiple, potentially overlapping, rewrite rules in parallel by identifying common sub-rules. For example, one could amalgamate two copies of the string graph rewrite rule



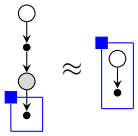
using the common sub-rule



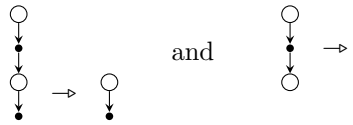
to produce the familier rule



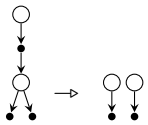
Doing this an arbitrary number of times would allow the construction of any one of the rules we intend



to represent. However, we cannot allow arbitrary amalgamations of this sort. While



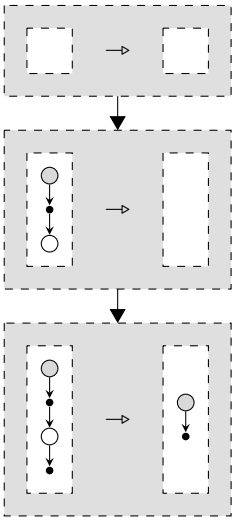
are both valid rules,



certainly is not.

Rensink's nested graph transformation rules[34, 35] extend this amalgamation idea using trees of rewrite rules, each rule being a subrule of its children. A predicate language over the left-hand side of these rules allows finer control of the matching process (and, by using a tree of graphs rather than of rewrite rules, the expression of complex graph predicates).

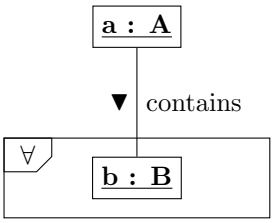
While the predicate language is unnecessary to codify spider-like equations, the idea of rule trees could be used as a way of encoding the !-graphs we described above, even including the nesting of !-boxes. For example, using Rensink's notation, we could describe the spidered version of X copies Z using the nested graph transformation rule



The fact that an *instance* of this tree can have any number of copies of any given branch (including none) gives us the family of rewrite rules we desire. However, this technique could not be used to encode the generalised bialgebra law, as the tree structure does not allow for overlapping !-boxes, and hence cannot express arbitrary bipartite graphs. The approach we take (encoding !-boxes in the

graph itself) also allows us to use graph rewriting to reason directly about !-graphs, as we will see in later chapters.

Stallmann produced an extension to the UML-based *Story Patterns* employed by FuJaBa[30] in [38]. These *enhanced Story Patterns* resemble Rensink's nested graph transformation rules in character, and resemble !-graphs in presentation. They allow parts of a Story Pattern to be marked with modifiers familiar from predicate logic, such as $\neg$, $\wedge$ or $\forall$. For example, the following enhanced Story Pattern captures the requirement that every instance of class B is contained in an instance of class A:



Chapter 5

!-Graph Rewriting

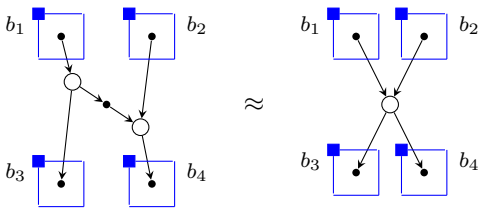
!-graphs provide the foundation for representing infinite families of string graph equations and rewrite rules. In this chapter, we will demonstrate the construction of equations and rewrite rules composed of !-graphs and show how they can be used to rewrite string graphs, producing new string graph equations, and even !-graphs, producing new !-graph equations. Importantly, the latter is sound with respect to the interpretation of !-graph equations as families of string graph equations.

The first two sections of this chapter are again based on [26] (pp 9-11). The other two sections have not previously been published.

5.1 !-Graph Equations

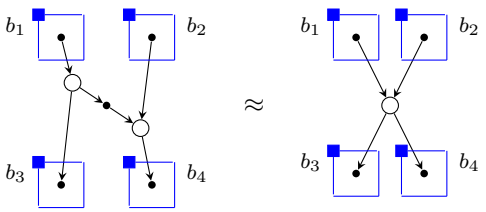
We can combine !-graphs to form !-graph equations in much the same way we did string graph equations. We will extend the !-box operations (and hence the idea of instantiation) to these equations, and this will determine the string graph equations represented by a given !-graph equation.

We can view a string graph equation as a pair of string graphs with correlated inputs and outputs. In the same way, we want a !-graph equation to encode a pair of !-graphs and a correlation of inputs, outputs and !-vertices. For example, the spider law given in (4.1) could be represented as



(5.1)

If we COPY_{b_2}, DROP_{b_3} and KILL_{b_4}, we get the following equation.



The rest of this section will be devoted to formalising these constructions and showing that we can apply the !-box operations to them.

We will use a span of monomorphisms $L \leftarrow I \rightarrow R$, where I equates $\mathrm{Bound}_!(L)$ with $\mathrm{Bound}_!(R)$. Before we get to the technical definition, we need a short lemma.

Lemma 5.1.1. *If G is a !-graph such that $U(G)$ has no isolated wire-vertices, then we have the following pushout:*

$$\begin{array}{ccc} \beta(G) & \hookrightarrow & \mathrm{In}_!(G) \\ \downarrow & & \downarrow \\ \mathrm{Out}_!(G) & \hookrightarrow & \mathrm{Bound}_!(G) \end{array}$$

Proof. If $U(G)$ has no isolated wire-vertices, $\mathrm{In}(U(G))$ and $\mathrm{Out}(U(G))$ are disjoint. The result then follows from the definitions and the fact that pushouts are unions in $\mathbf{Graph}/\mathcal{G}_{T!}$. □

The definition of a !-graph equation we give here looks quite different from the definition of a *rewrite pattern* given in [26] (pp 9). We are using this formulation to better demonstrate the similarities with string graph equations (definition 3.5.1), and because it is easier to use when constructing proofs.

Definition 5.1.2 (!-graph equation)**.** A !*-graph equation* $L \approx_{i_1,i_2} R$ is a span $L \xleftarrow{i_1} I \xrightarrow{i_2} R$ in $\mathbf{BGraph}_T$ where

1. $U(L)$ and $U(R)$ contain no isolated wire-vertices;
2. $\mathrm{In}_!(L) \cong \mathrm{In}_!(R)$ and $\mathrm{Out}_!(L) \cong \mathrm{Out}_!(R)$;
3. $\mathrm{Bound}_!(L) \cong I \cong \mathrm{Bound}_!(R)$; and
4. the following diagram commutes, where j_1, j_2, k_1 and k_2 are the inclusions from lemma 5.1.1 composed with the above isomorphisms:

$$\begin{array}{ccccc} & & \mathrm{In}_!(L) & \xrightarrow{\cong} & \mathrm{In}_!(R) & & \\ & \swarrow & & \searrow\ \ \swarrow & & \searrow & \\ L & \xleftarrow{i_1} & & I & & \xrightarrow{i_2} & R \\ & \searrow & & \nearrow\ \ \searrow & & \nearrow & \\ & & \mathrm{Out}_!(L) & \xrightarrow[\cong]{} & \mathrm{Out}_!(R) & & \end{array} \qquad (5.2)$$

A !*-graph rewrite rule* is a directed !-graph equation.

It can be seen by comparing the definitions that applying U to a !-graph equation will yield a string graph equation, and similarly for rewrite rules.

Note that the isomorphisms in the above definition are forced to agree with i_1 and i_2. In particular, lemma 5.1.1 can be used to show that the following diagram commutes

$$\begin{array}{ccccc} \mathrm{Bound}_!(L) & \xrightarrow{\cong} & I & \xleftarrow{\cong} & \mathrm{Bound}_!(R) \\ \downarrow & \swarrow^{i_1} & & \searrow_{i_2} & \downarrow \\ L & & & & R \end{array} \tag{5.3}$$

and if we expand the definitions of j_1, j_2, k_1 and k_2 in (5.2), we get

$$\begin{array}{ccccc} \mathrm{In}_!(L) & & \xrightarrow{\quad\cong\quad} & & \mathrm{In}_!(R) \\ \downarrow & & & & \downarrow \\ \mathrm{Bound}_!(L) & \xrightarrow{\cong} & I & \xleftarrow{\cong} & \mathrm{Bound}_!(R) \\ \uparrow & & & & \uparrow \\ \mathrm{Out}_!(L) & & \xrightarrow{\quad\cong\quad} & & \mathrm{Out}_!(R) \end{array} \tag{5.4}$$

We will now extend the !-box operations to !-graph equations. The intuitive rule is that the same operation must be performed on all three graphs in the span, with the correspondence between !-boxes determined by the morphisms of the span. Of course, to get an equation, we need some way of updating the morphisms of the span in addition to the graphs.

It turns out we can do this with any monomorphism of !-graphs that reflects !-box containment:

Lemma 5.1.3. *Let $f : G \to H$ be a !-graph monomorphism that reflects !-box containment, and let b be a !-vertex in G. Then there are !-graph monomorphisms*

$$\mathrm{DROP}_{f(b)}(f) : \mathrm{DROP}_b(G) \to \mathrm{DROP}_{f(b)}(H)$$
$$\mathrm{KILL}_{f(b)}(f) : \mathrm{KILL}_b(G) \to \mathrm{KILL}_{f(b)}(H)$$
$$\mathrm{COPY}_{f(b)}(f) : \mathrm{COPY}_b(G) \to \mathrm{COPY}_{f(b)}(H)$$

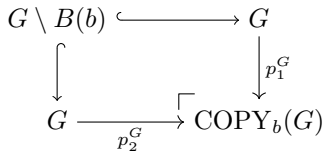
that reflect !-box containment and that commute with f in the following ways:

$$\begin{array}{ccc} \mathrm{DROP}_b(G) & \xrightarrow{\mathrm{DROP}_{f(b)}(f)} & \mathrm{DROP}_{f(b)}(H) \\ \downarrow & & \downarrow \\ G & \xrightarrow{\quad f \quad} & H \end{array} \tag{5.5}$$

$$\begin{array}{ccc} \mathrm{KILL}_b(G) & \xrightarrow{\mathrm{KILL}_{f(b)}(f)} & \mathrm{KILL}_{f(b)}(H) \\ \downarrow & & \downarrow \\ G & \xrightarrow{\quad f \quad} & H \end{array} \tag{5.6}$$

$$\begin{array}{ccc} \mathrm{COPY}_b(G) & \xrightarrow{\mathrm{COPY}_{f(b)}(f)} & \mathrm{COPY}_{f(b)}(H) \\ {\scriptstyle p_i^G}\uparrow & & \uparrow{\scriptstyle p_i^H} \\ G & \xrightarrow{\quad f \quad} & H \end{array} \tag{5.7}$$

where $i \in \{1, 2\}$ and p_1^G and p_2^G are the maps from the pushout defining $\mathrm{COPY}_b(G)$*:*



and p_1^H and p_2^H are defined similarly.

These maps are all the unique ones satisfying these diagrams, up to unique isomorphism.

Proof. Let $G' = \mathrm{DROP}_b(G)$ and $H' = \mathrm{DROP}_{f(b)}(H)$. Note that the image of G' under f is contained in H', since f is injective (so no vertex of G other than b maps to $f(b)$). So there is a unique monomorphism $f' = \mathrm{DROP}_{f(b)}(f)$ satisfying (5.5). f' reflects !-box containment, since if $f'(v) \in B_H(c)$, $f(v) \in B_{H'}(c)$, and $c \neq b$.

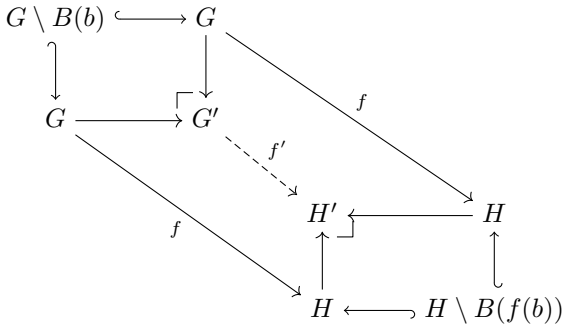
Now we deal with $\mathrm{KILL}_{f(b)}(f)$. Let $G' = \mathrm{KILL}_b(G)$ and $H' = \mathrm{KILL}_{f(b)}(H)$. We start by showing that the image of G' under f is contained in H'. These are full subgraphs of G and H, respectively, so we only need to consider vertices. Let v be a vertex in G'. So v is not in $B(b)$. Then, since f reflects !-box containment, $f(v)$ is not in $B(f(b))$, and so is in H'. So we can construct the monomorphism $f' = \mathrm{KILL}_{f(b)}(f)$ by simply restricting the domain and codomain of f. Since the inclusion of $\mathrm{KILL}_{f(b)}(H)$ into H is monic, this morphism must be the unique one satisfying (5.6).

Now we show that f' reflects !-box containment. Let v be a vertex of G' and $c \in !(H')$, with $f'(v) \in B_{H'}(c)$. Then $f(v) \in B_H(c)$, and so c and the edge from c to $f(v)$ are both in the image of f. Further, c cannot be in $B_H(f(b))$, since it is in H'. So the preimage of c' cannot be in $B_G(b)$, and hence must be in G', along with the joining edge, and so c and the edge from c to v are in the image of f'.

Finally, we consider $\mathrm{COPY}_{f(b)}(f)$. Let $G' = \mathrm{COPY}_b(G)$ and $H' = \mathrm{COPY}_{f(b)}(H)$. We demonstrate the existance of $\mathrm{COPY}_{f(b)}(f)$ by pushout. If we let $\iota_G : G \setminus B(b) \to G$ and $\iota_H : H \setminus B(f(b)) \to H$ be the natural inclusion maps, (5.6) gives us that

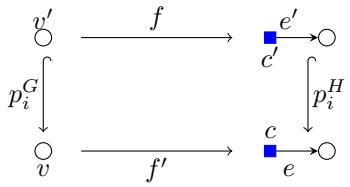
$$p_1^H \circ f \circ \iota_G = p_1^H \circ \iota_H \circ \mathrm{KILL}_{f(b)}(f) = p_2^H \circ \iota_H \circ \mathrm{KILL}_{f(b)}(f) = p_1^H \circ f \circ \iota_G$$

and there is then a unique f' making the following diagram commute:



Since all the other arrows in the diagram are monic, f' must be as well. For a given choice of p_1^H and p_2^H (relative to p_1^G and p_2^G), it is the unique morphism satisfying (5.7). If we do swap p_1^H and p_2^H, we can note that the pushout is symmetric, and pushouts are unique up to a unique isomorphism, so f' is unique up to a unique isomorphism.

We must show that f' reflects !-box containment. Suppose v is a vertex in G' and $f(v) \in B(c)$, where c is a !-vertex in H', with e the edge from c to $f(v)$. e is in the image of one of the p_i^H; we call the preimage of e under it e', and its source (which maps to c) c'. $f'(v)$ must also be in the image of i_H, and its preimage must be in the image of f. What is more, the preimage of this under f must map to v by p_i^G, due to (5.7).



We know that, since f reflects !-box containment, e' must be in the image of f. But this means that e must be in the image of f', and hence f' also reflects !-box containment. So then we just let $\mathrm{COPY}_{f(b)}(f) = f'$. □

Given a !-graph equation $L \approx_{i_1,i_2} R$, we know that i_1 and i_2 are isomorphisms when their codomains are restricted to $\mathrm{Bound}_!(L)$ and $\mathrm{Bound}_!(R)$, respectively. This means that they must reflect !-box containment, so we can now make the following definitions.

Definition 5.1.4 ([26], pp 10)**.** Let $L \approx R$ be a !-graph equation defined by the span $L \xleftarrow{i_1} I \xrightarrow{i_2} R$, and let $b \in !(I)$, with $b_1 = i_1(b)$ and $b_2 = i_2(b)$. Then the !-box operations on !-graphs have the following equivalents on !-graph equations:

- $\mathrm{COPY}_b(L \approx R)$ is defined to be

$$\mathrm{COPY}_{b_1}(L) \xleftarrow{\mathrm{COPY}_{b_1}(i_1)} \mathrm{COPY}_b(I) \xrightarrow{\mathrm{COPY}_{b_2}(i_2)} \mathrm{COPY}_{b_2}(R)$$

- $\mathrm{DROP}_b(L \approx R)$ is the span

$$\mathrm{DROP}_{b_1}(L) \xleftarrow{\mathrm{DROP}_{b_1}(i_1)} \mathrm{DROP}_b(I) \xrightarrow{\mathrm{DROP}_{b_2}(i_2)} \mathrm{DROP}_{b_2}(R)$$

- $\mathrm{KILL}_b(L \approx R)$ is the span

$$\mathrm{KILL}_{b_1}(L) \xleftarrow{\mathrm{KILL}_{b_1}(i_1)} \mathrm{KILL}_b(I) \xrightarrow{\mathrm{KILL}_{b_2}(i_2)} \mathrm{KILL}_{b_2}(R)$$

In the case of !-graph rewrite rules, the direction is preserved in the obvious way.

Note that the equivalent definitions in [26] are the same; the definition here appears simpler because we have made use of lemma 5.1.3.

We must, of course, ensure that these operations produce !-graph equations. We start with some useful results about the operations of lemma 5.1.3.

Lemma 5.1.5. *All the operations from lemma 5.1.3 respect composition, at least up to unique isomorphism. For example,*

$$\mathrm{DROP}_{g(f(b))}(g \circ f) = \mathrm{DROP}_{g(f(b))}(g) \circ \mathrm{DROP}_{f(b)}(f)$$

Proof. Consider the case of DROP. Suppose we have !-box-reflecting monomorphisms

$$f : G_1 \to G_2$$
$$g : G_2 \to G_3$$

and suppose $b \in G_1$. Let $f' = \mathrm{DROP}_{f(b)}(f)$, $G_1' = \mathrm{DROP}_b(G_1)$ and so on. Then we just consider the diagram

$$\begin{array}{ccccc} G_1' & \xrightarrow{f'} & G_2' & \xrightarrow{g'} & G_3' \\ \downarrow & & \downarrow & & \downarrow \\ G_1 & \xrightarrow{f} & G_2 & \xrightarrow{g} & G_3 \end{array}$$

But $\mathrm{DROP}_{g(f(b))}(g \circ f)$ is the unique map from G_1' to G_3' that makes the outside edges of that diagram commute, and so must be the same as $g' \circ f'$. The same argument holds for $\mathrm{KILL}_{g(f(b))}(g \circ f)$.

A similar argument can be used for COPY, but care must be taken, as the result of applying COPY to a map is only unique up to isomorphism. It is possible, however, to make the relevant choices such that $\mathrm{COPY}_{g(f(b))}(g \circ f) = \mathrm{COPY}_{g(f(b))}(g) \circ \mathrm{COPY}_{f(b)}(f)$, which is sufficient to get the result we want. □

Lemma 5.1.6. *All the operations from lemma 5.1.3 preserve isomorphisms.*

Proof. This follows from lemma 5.1.5 and the fact that $\mathrm{DROP}_b(1_G)$ and $\mathrm{KILL}_b(1_G)$ are the identity, and $\mathrm{COPY}_b(1_G)$ is an isomorphism of $\mathrm{COPY}_b(G)$. □

Lemma 5.1.7. *Let G be a graph with no isolated vertices. Then $\mathrm{COPY}_b(G)$, $\mathrm{KILL}_b(G)$ and $\mathrm{DROP}_b(G)$ do not have any isolated vertices.*

Proof. If w were an isolated wire-vertex in $U(\mathrm{COPY}_b(G))$, it must be the image of a map from $U(G)$, but $U(G)$ does not contain any isolated wire-vertices, and so the map must carry an incident edge with it.

$U(\mathrm{KILL}_b(G))$ is a full subgraph of $U(G)$ and $U(B(b))$ is not adjacent to any wire-vertices, so any wire-vertex in $U(G')$ must have the same incident edges as the corresponding wire-vertex in $U(G)$.

$U(\mathrm{DROP}_b(G))$ is isomorphic to $U(G)$, and so has no isolated wire-vertices. □

Theorem 5.1.8. *Let $L \approx R$ be a !-graph equation. Then applying any of the !-box operations of definition 5.1.4 yields another !-graph equation.*

Proof. Let $L \xleftarrow{i_1} I \xrightarrow{i_2} R$ be a !-graph equation, and $L' \xleftarrow{i'_1} I' \xrightarrow{i'_2} R'$ be the result of applying one of the !-box operations to it. We will demonstrate the proof for COPY_b, but all of the arguments apply to KILL_b and DROP_b as well.

It is clear from lemma 5.1.3 that this is a span of !-graphs.

$U(L')$ and $U(R')$ do not contain any isolated wire-vertices, by lemma 5.1.7.

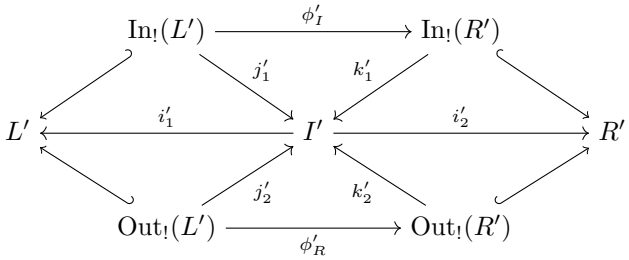
Since $L \xleftarrow{i_1} I \xrightarrow{i_2} R$ is a !-graph equation, we have isomorphisms

$$\phi_I : \mathrm{In}_!(L) \to \mathrm{In}_!(R)$$
$$\phi_O : \mathrm{Out}_!(L) \to \mathrm{Out}_!(R)$$
$$\phi_L : \mathrm{Bound}_!(L) \to I$$
$$\phi_R : \mathrm{Bound}_!(R) \to I$$

These all preserve !-box containment, and hence we have isomorphisms (by lemma 5.1.6 and theorem 4.3.7)

$$\phi'_I : \mathrm{In}_!(L') \to \mathrm{In}_!(R')$$
$$\phi'_O : \mathrm{Out}_!(L') \to \mathrm{Out}_!(R')$$
$$\phi'_L : \mathrm{Bound}_!(L') \to I'$$
$$\phi'_R : \mathrm{Bound}_!(R') \to I'$$

Finally, we need to show that the following diagram commutes:

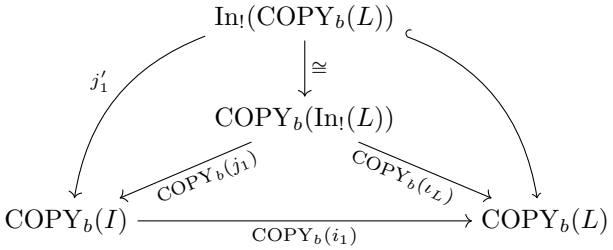


(5.8)

where j'_1 is ϕ'_L composed with the inclusion of $\mathrm{In}_!(L')$ into $\mathrm{Bound}_!(R')$, and similarly for j'_2, k'_1 and k'_2.

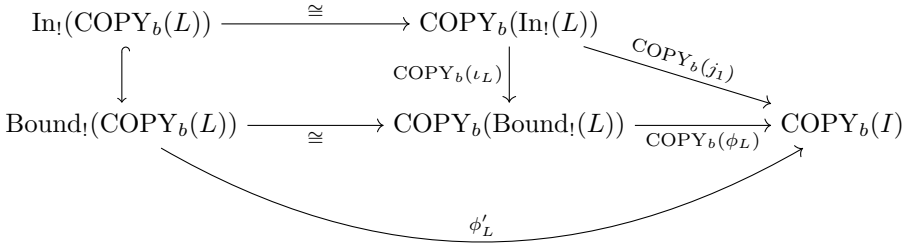
The proofs for the four triangles on the left and right are all similar, so we will demonstrate the

top left triangle. Consider the following diagram:



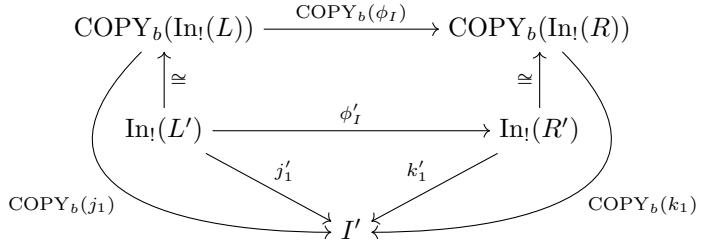
(5.9)

where $\iota_L : \text{In}_!(L) \to L$ is the normal inclusion map. The bottom triangle commutes by lemma 5.1.5, since $\iota_L = i_1 \circ j_1$ by our assumption that $L \xleftarrow{i_1} I \xrightarrow{i_2} R$ is a !-graph equation. That the upper right triangle commutes can be seen from the proof of theorem 4.3.7. For the upper left triangle, consider the following:



where we have abused ι_L to refer to the inclusion of $\text{In}_!(L)$ into $\text{Bound}_!(L)$. Now we can again use the proof of theorem 4.3.7 to see that the square on the left commutes, the triangle on the right commutes by lemma 5.1.5 and the definition of j_1, and the bottom triangle is simply the definition of ϕ_L'. But now j_1' is ϕ_L' composed with the inclusion on the left of the diagram, by definition, and so we have that the top left triangle of (5.9) commutes, and hence the upper left triangle of (5.8) commutes.

We now look at the upper middle triangle of (5.8); the lower middle triangle follows similarly. Consider the following diagram:



We have already demonstrated that the triangle on the left commutes, and the triangle on the right follows in a similar manner. The top square is just the definition of ϕ_I', and the outside edges of the diagram commute by lemma 5.1.5 and the assumption that $L \xleftarrow{i_1} I \xrightarrow{i_2} R$ is a !-graph equation. It follows that the triangle at the bottom commutes, and this is the top middle triangle of (5.8).

Hence we have shown that $L' \xleftarrow{i_1'} I' \xrightarrow{i_2'} R'$ is, indeed, a !-graph equation. □

Corollary 5.1.9 ([26], pp 11)**.** *Let $L \twoheadrightarrow R$ be a !-graph rewrite rule. Then applying any of the !-box operations of definition 5.1.4 yields another !-graph rewrite rule.*

The concept of instantiation from definition 4.4.1 then extends to !-graph equations and rewrite rules in the obvious way. We will consider a !-graph equation to hold if and only if all its concrete instantiations hold.

5.2 Rewriting String Graphs with !-Graph Equations

For a !-graph equation $L \approx R$ and a string graph G, we want to use the directed form of the equation, $L \twoheadrightarrow R$, to rewrite G, and the resulting graph should be a valid rewrite of G in the (usually infinite) string graph rewrite system comprised of the concrete instantiations of $L \twoheadrightarrow R$.

Because applying a !-box operation to a !-graph rewrite rule also applies it to the components of that rule, and the LHS in particular, we can use the notion of matching from section 4.4 to find a concrete instance of the rewrite rule that can be applied to a given string graph G.

Suppose we have a !-graph rewrite rule $L \twoheadrightarrow R$ and a string graph G. If m is a matching of L onto G under S, we can apply S to $L \twoheadrightarrow R$ to get a string graph rewrite rule $L' \twoheadrightarrow R'$. By considering definition 5.1.4, we can see that L' is the string graph obtained by instantiating L with S. So we have a matching from L' to G, and can therefore rewrite G with $L' \twoheadrightarrow R'$.

Providing finding the instantiation and matching is decidable (which we demonstrate in chapter 7), rewriting G with $L \twoheadrightarrow R$ is decidable, and our requirement that this is a valid rewrite under the rewrite system of instantiations of $L \twoheadrightarrow R$ is satisfied.

So we have acheived our aim of expressing certain infinite families of string graph rewrite rules in a finite (and useful) way. In the next section, we will consider how !-graph rewrite rules can actually be used to rewrite !-graphs, and thereby derive new !-graph rewrite rules.

5.3 Rewriting !-Graphs With !-Graph Equations

Let $L \xrightarrow{m} G$ be a !-graph monomorphism. Then we have the following pushout complement

$$\begin{array}{ccc} \mathrm{Bound}_!(L) & \hookrightarrow & L \\ {\scriptstyle d}\downarrow & & \downarrow{\scriptstyle m} \\ D & \xrightarrow[g]{} & G \end{array} \tag{5.10}$$

in $\mathbf{Graph}/\mathcal{G}_{T!}$ if and only if m satisfies the no-dangling-edges condition. This condition is that for every vertex v in $L \setminus \mathrm{Bound}_!(L)$, and every edge e in G incident to $m(v)$, e is in the image of m. This is stronger than $U(m)$ being a local isomorphism, as it also requires that if a !-vertex b and a vertex v in $L \setminus \mathrm{Bound}_!(L)$ are both in the image of m, any edge between them must also be in the image of m.

Proposition 5.3.1. *If the pushout complement* (5.10) *exists, D is a* !*-graph.*

Proof. As all the morphisms in the diagram are monic, we consider D and L to be subgraphs of G, and $\text{Bound}_!(L)$ their common subgraph.

Consider applying U to the pushout square. This is a pushout in $\mathbf{Graph}/\mathcal{G}_T$, by proposition 3.3.4. But pushout complements are unique up to isomorphism in $\mathbf{Graph}/\mathcal{G}_T$, and we know that the pushout complement of

$$\text{Bound}(U(L)) \hookrightarrow U(L) \xrightarrow{U(m)} U(G)$$

is a string graph (since $U(m)$ is a local isomorphism, by the no-dangling-edges condition), so $U(D)$ must be a string graph.

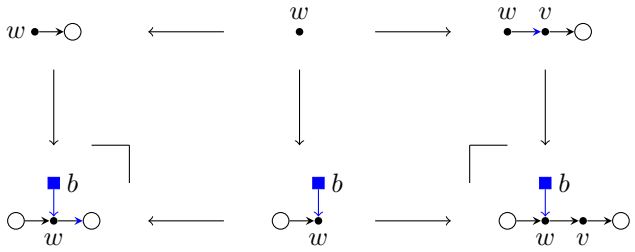
We apply β to the pushout square, giving us another pushout (again, by proposition 3.3.4). Then, since $\beta(\text{Bound}_!(L)) = \beta(L)$, we must have that $\beta(D) \cong \beta(G)$, and hence $\beta(D)$ must be posetal.

Let $b \in !(D)$. Then $b \in !(G)$. We know that $U(B_G(b))$ is open in $U(G)$, and $B_D(b) = B_G(b) \cap D$, so $U(B_D(b)) = U(B_G(b)) \cap U(D)$. But this is open in $U(D)$ by proposition 4.1.3.

Let $b, c \in !(D)$, with $c \in B_D(b)$, and let $v \in B_D(c)$. Then $b, c \in !(G)$ and $v \in B_G(c)$, and so $v \in B_G(b)$. The edge linking b to v must either be in D or L. suppose it is in L. Then so must b and v be, and so b and v must be in $\text{Bound}_!(L)$. But then the edge linking them must also be in $\text{Bound}_!(L)$ (by definition of $\text{Bound}_!$) and hence in D. So $v \in B_D(b)$.

So D is a !-graph. □

Note that, while the no-dangling-edges condition will always produce a pushout complement, it will not always lead to a valid rewrite. For example, in the following commutative diagram in $\mathbf{Graph}/\mathcal{G}_{T!}$, the map on the left satisfies the no-dangling-edges condition, but the resulting (bottom right) graph is not a valid !-graph, as $B(b)$ is not an open subgraph.



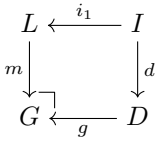
(5.11)

Definition 5.3.2 (!-graph matching)**.** A monomorphism $L \xrightarrow{m} G$ between two !-graphs is a !*-graph matching* when $U(m)$ is a local isomorphism and m reflects !-box containment. If $L_0 \succeq L$, with instantiation S, m is said to be a !-graph matching of L_0 onto G under S.

Of course, if m is a !-graph matching, then m and the inclusion of $\text{Bound}_!(L)$ in L have a pushout complement. But, unlike in (5.11), a !-graph matching always produces a rewrite.

Theorem 5.3.3. *Let $L \rightarrowtriangle R$ be a* !*-graph rewrite rule and $m : L \to G$ be a* !*-graph matching for it. Then $L \rightarrowtriangle R$ has a rewrite at m.*

Proof. Let $L \xleftarrow{i_1} I \xrightarrow{i_2} R$ be the !-graph rewrite rule, and let

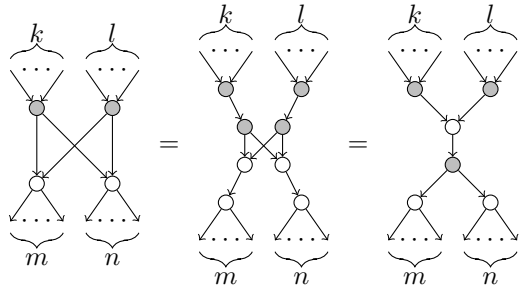


be the pushout complement of i_1 and m. Then it is sufficient (by proposition 4.3.3) to show that the span formed from d and i_2 is boundary-!-coherent.

Theorem 3.6.7 gives us that $U(i_1)$ and $U(d)$ are boundary-coherent, and then lemma 3.6.8 gives us that $U(d)$ and $U(i_2)$ must also be boundary-coherent.

i_2 reflects !-box containment by the definition of a !-graph rewrite rule. Let e be an edge in D from a !-vertex b to a vertex v, where v is in the image of d. Then $g(v)$ must be in the image of m, and so $g(e)$ must be in the image of m, since m reflects !-box containment. But then, since this is a pushout square, e must be in the image of d as required, and so d also reflects !-box containment. Thus d and i_2 are boundary-!-coherent, as required. □

Example 5.3.4. Recall the following proof from section 2.2:

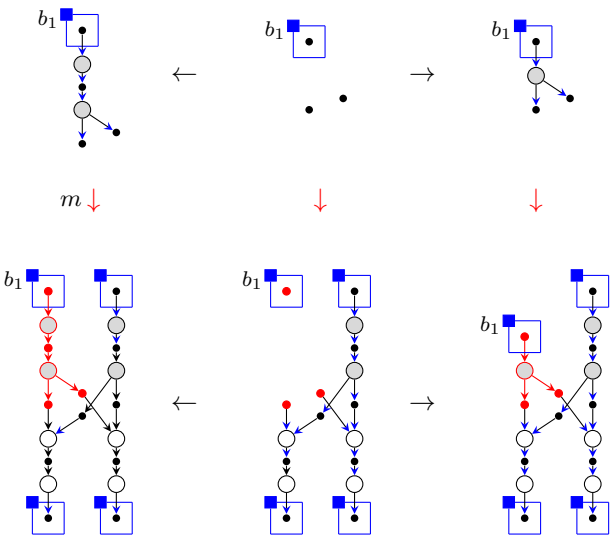


Consider the left equality. We can perform the equivalent rewrite by using instances of the X and Z spider laws

These instances are

We can use the first of these to construct the following rewrite (ignoring issues of wire homeomor-

phism):



$U(m)$ is a local isomorphism, and m can be seen to reflect !-box containment (for b_1, in this case). As expected, we get a valid !-graph at the end. One more application of the X spider law and two of the Z spider law will produce the chain of rewrites we want.

Suppose we have the following rewrite (where m is not necessarily a !-graph matching):

$$\begin{array}{ccccc} L & \xleftarrow{i_1} & I & \xrightarrow{i_2} & R \\ {\scriptstyle m}\downarrow & & \downarrow{\scriptstyle d} & & \downarrow{\scriptstyle r} \\ G & \xleftarrow[g]{} & D & \xrightarrow[h]{} & H \end{array} \tag{5.12}$$

The bottom span will almost certainly not be a !-graph rewrite rule (as $U(D)$ will not, in general, have only isolated points), but it is trivial to produce such a rewrite rule by discarding unwanted vertices and edges from D.

Theorem 5.3.5. *In* (5.12)*, if* $U(G)$ *has no isolated points then there is a subgraph* J *of* D *such that*

$$G \xleftarrow{g\restriction_J} J \xrightarrow{h\restriction_J} H$$

is a !*-graph rewrite rule.*

Proof. We start by showing that $\text{Bound}_!(G)$ is in the image of g.

$\beta(i_1)$ and $\beta(i_2)$ are isomorphisms since i_1 and i_2 form a !-graph rewrite rule. It then follows from the pushout squares that $\beta(g)$ and $\beta(h)$ are also isomorphisms. So we must have that $\beta(G)$ is in the image of g.

If w is in the boundary of $U(G)$, it is either not in the image of m, in which case it must be in the image of g, or its preimage under m is in the boundary of $U(L)$ (since string graph monomorphisms reflect boundaries), in which case this preimage is in the image of i_1 and hence w is in the image of g. Now if $b \in !(G)$ and there is an edge e from b to w in G, we also know that if e is not in the image

of m, it must be in the image of g. So suppose e is in the image of m. Then its preimage must be in the image of i_1 by the same reasoning as above, and hence e is in the image of g. So $\text{Bound}_!(G)$ is in the image of g.

We can use the same argument to show that $\text{Bound}_!(H)$ is in the image of h.

Let J be the preimage of $\text{Bound}_!(G)$ under g, and $\iota : J \hookrightarrow D$ the inclusion of J in D. $g \circ \iota$ is monic, and hence injective, so we have $J \cong \text{Bound}_!(G)$.

We already know that the image of $\beta(J)$ under $h \circ \iota$ is $\beta(H)$, since $\beta(\iota)$ and $\beta(h)$ are both isomorphisms. We now show that, for all vertices w in J, $h(\iota(w))$ is an input (resp. output) if and only if $g(\iota(w))$ is an input (resp. output). This will then give us that the image of $h \circ \iota$ is exactly $\text{Bound}_!(H)$ (since we already know that $\text{Bound}_!(H)$ is in that image).

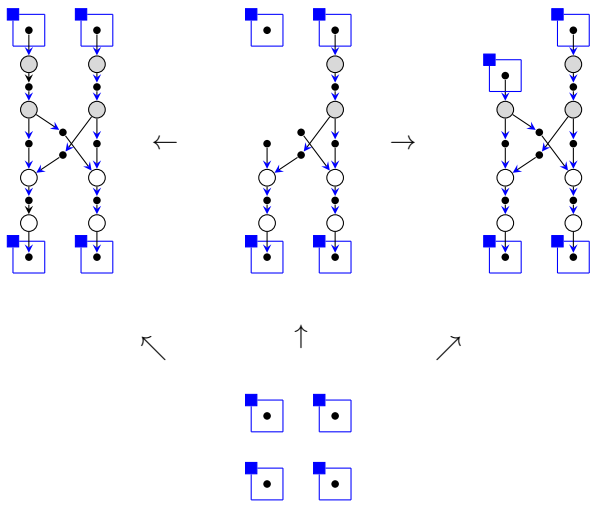
So let w be a wire-vertex in J. We know that $g(\iota(w))$ is either an input or an output. Suppose it is an input (the output case is symmetric). Then $\iota(w)$ must also be an input, and if $g(\iota(w))$ is in the image of m, its preimage must be an input. Let w' be the preimage of $g(\iota(w))$ under $m \circ i_1$. Then we know that $d(w') = \iota(w)$ and hence $r(i_2(w')) = h(\iota(w))$, and $i_2(w')$ is an input (since $i_1(w')$ is). So the preimages of $h(\iota(w))$ under both h and r are inputs, and hence $h(\iota(w))$ must also be an input. If $g(\iota(w))$ is not in the image of m, then $\iota(w)$ is not in the image of d, and hence $h(\iota(w))$ is not in the image of r. Thus r cannot introduce an incoming edge for $h(\iota(w))$, which must therefore be an input.

Injectivity of $h \circ \iota$ gives us the isomorphism $J \cong \text{Bound}_!(H)$. This restricts to $\text{In}_!(G) \cong \text{In}_!(H)$ and $\text{Out}_!(G) \cong \text{Out}_!(H)$, as we just proved that $h(\iota(w))$ is an input if and only if $g(\iota(w))$ is an input. The diagram from definition 5.1.2 commutes by construction, and hence

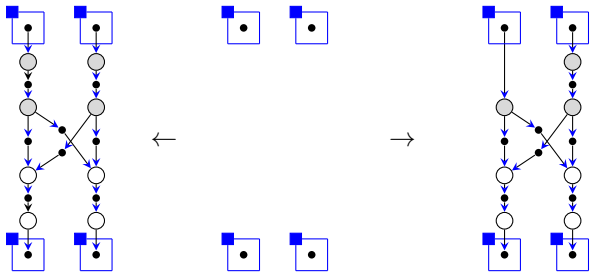
$$G \xleftarrow{g \circ \iota} J \xrightarrow{h \circ \iota} H$$

is a !-graph rewrite rule. □

Example 5.3.6. Continuing example 5.3.4, we get the following commutative diagram:



Thus we have the !-graph rewrite rule



5.4 Soundness of !-Graph Rewriting

We have shown how we can rewrite !-graphs using !-graph rewrite rules, and in doing so produce further !-graph rewrite rules. However, we also want to ensure that the resulting rewrite rule is sound with respect to its interpretation as a family of string graph rewrite rules.

Note that this differs from soundness of string graph equations or rewrite rules, as discussed in chapter 3, where soundness was with respect to a particular valuation. Instead, we are interested in soundness with respect to a particular rewrite sytem of string graphs.

Specifically, if we have a !-graph rewrite rule $L \twoheadrightarrow R$ which has been used to rewrite a !-graph L' to another !-graph R', and there is an instance of the resulting !-graph rewrite rule $L' \twoheadrightarrow R'$ that can rewrite the string graph G to H, we require there to be an instance of $L \twoheadrightarrow R$ that also rewrites G to H.

Luckily, not only does our definition of !-graph matching guarantee that we get a rewrite, it also ensures that we can preserve the rewrite under !-box operations. We know that the top span (L, I and R) share !-vertices (in that we get isomorphisms if we apply β to the maps), and the same is true of the bottom span (G, D and H; see the proof of theorem 5.3.5). For notational convenience, we will identify the !-vertices across each of these spans (eg: if b is a !-vertex in I, we will consider b to be in L as well, by which we mean $i_1(b)$).

Further, because the maps m, d and r of (5.12) are monic, we can identify each !-vertex in the top span with one in the bottom span. So, given that we wish to perform a !-box operation (which we will denote OP_b) on G and propagate it to the entire rewrite, we have two possible cases: either the !-vertex being operated on is in the image of m, in which case we consider it as existing across the whole rewrite, or it is not, in which case it is only in the bottom span.

If b is in the image of m, lemma 5.1.3 provides us with a way of updating all the maps of the rewrite:

$$\begin{array}{ccccc} \mathrm{OP}_b(L) & \xleftarrow{\mathrm{OP}_b(i_1)} & \mathrm{OP}_b(I) & \xrightarrow{\mathrm{OP}_b(i_2)} & \mathrm{OP}_b(R) \\ \downarrow{\scriptstyle \mathrm{OP}_b(m)} & & \downarrow{\scriptstyle \mathrm{OP}_b(d)} & & \downarrow{\scriptstyle \mathrm{OP}_b(r)} \\ \mathrm{OP}_b(G) & \xleftarrow[\mathrm{OP}_b(g)]{} & \mathrm{OP}_b(D) & \xrightarrow[\mathrm{OP}_b(h)]{} & \mathrm{OP}_b(H) \end{array} \tag{5.13}$$

We will, of course, need to check that this is still a rewrite.

If b is not in the image of m, lemma 5.1.3 is insufficient. However, the following lemma will allow us to extend !-box operations on maps to !-vertices that are not in the image of the map.

Lemma 5.4.1. *Let $f : G \to H$ be a !-graph monomorphism that reflects !-box containment, and let b be a !-vertex in H but not in the image of f. Then there are !-graph monomorphisms*

$$\mathrm{DROP}_b(f) : G \to \mathrm{DROP}_b(H)$$
$$\mathrm{KILL}_b(f) : G \to \mathrm{KILL}_b(H)$$
$$\mathrm{COPY}_b(f) : G \to \mathrm{COPY}_b(H)$$

that reflect !-box containment and that commute with f in the following ways:

$$\begin{array}{ccc} G & \xrightarrow{\mathrm{DROP}_b(f)} & \mathrm{DROP}_b(H) \\ & \searrow^{f} & \downarrow \\ & & H \end{array} \tag{5.14}$$

$$\begin{array}{ccc} G & \xrightarrow{\mathrm{KILL}_b(f)} & \mathrm{KILL}_b(H) \\ & \searrow^{f} & \downarrow \\ & & H \end{array} \tag{5.15}$$

$$\begin{array}{ccc} G & \xrightarrow{\mathrm{COPY}_b(f)} & \mathrm{COPY}_b(H) \\ & \searrow^{f} & \uparrow{\scriptstyle p_i^H} \\ & & H \end{array} \tag{5.16}$$

where $i \in \{1, 2\}$ and p_1^H and p_2^H are the maps from the pushout that defines $\mathrm{COPY}_b(H)$.

Proof. Since b is not in the image of f (and so neither are any of its edges), we let $\mathrm{DROP}_b(f)$ be f, but with codomain $\mathrm{DROP}_b(H)$. Simply restricting the codomain cannot break the property of reflecting !-box containment, so this is inherited from f.

$\mathrm{KILL}_b(f)$ follows in the same way once we note that, since f reflects !-box containment, the image of f cannot intersect $B(b)$.

Recall the definition of $\mathrm{COPY}_b(H)$:

$$\begin{array}{ccc} H \setminus B(b) & \longrightarrow & H \\ \downarrow & & \downarrow{\scriptstyle p_1^H} \\ H & \xrightarrow[p_2^H]{} & \mathrm{COPY}_b(H) \end{array}$$

Let ι be the induced inclusion of $H \setminus B(b)$ into $\mathrm{COPY}_b(H)$ (equal to both composed maps in the diagram).

Now we define

$$\mathrm{COPY}_b(f) := \iota \circ \mathrm{KILL}_b(f)$$

and so we have a monomorphism that satisfies (5.16). For the remainder of this proof, we will use f' to refer to $\mathrm{COPY}_b(f)$ for convenience.

Let v be a vertex in G and e an edge from a !-vertex b to $f'(v)$ in $\mathrm{COPY}_b(H)$. e must be in the image of at least one of the p_i^H. Then (one of) its preimage(s) e' is an edge from a !-vertex to $f(v)$ in H, and so e' must be in the image of f (as f reflects !-box containment). So e is in the image of f', and hence f' also reflects !-box containment. □

Armed with this result, if b is not in the image of m, we can do the following:

$$\begin{array}{ccccc} L & \xleftarrow{i_1} & I & \xrightarrow{i_2} & R \\ {\scriptstyle \mathrm{OP}_b(m)}\downarrow & & \downarrow{\scriptstyle \mathrm{OP}_b(d)} & & \downarrow{\scriptstyle \mathrm{OP}_b(r)} \\ \mathrm{OP}_b(G) & \xleftarrow[\mathrm{OP}_b(g)]{} & \mathrm{OP}_b(D) & \xrightarrow[\mathrm{OP}_b(h)]{} & \mathrm{OP}_b(H) \end{array} \tag{5.17}$$

which we again need to show is a rewrite. The first step (for both (5.13) and (5.17)) is to show that the !-box operations preserve the property of being a !-graph matching.

Lemma 5.4.2. *Let $f : G \to H$ be a !-graph matching and let b be a !-vertex in H. Then* $\mathrm{DROP}_b(f)$, $\mathrm{KILL}_b(f)$ *and* $\mathrm{COPY}_b(f)$ *are all !-graph matchings.*

Proof. Lemmas 5.4.1 and 5.1.3 give us that they are monomorphisms that reflect !-box containment. So we just need to show that they are local isomorphisms.

$U(\mathrm{DROP}_b(f)) = U(f)$, which we already know to be a local isomorphism.

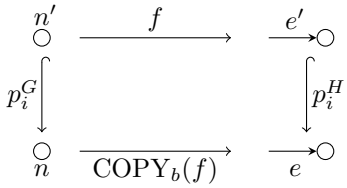
If b is not in the image of f, we note that since f reflects !-box containment, the image of f cannot intersect $B(b)$. Then, since simply restricting the codomain cannot break the property of being a local isomorphism, $U(\mathrm{KILL}_b(f))$ must be a local isomorphism, since $U(f)$ is.

Otherwise, suppose b is in the image of f, and $U(\mathrm{KILL}_b(f))$ is not a local isomorphism. Let n be a node-vertex in the image of $\mathrm{KILL}_b(f)$ with an incident edge e not in its image. We know that e is in the image of f; let its preimage be e'. Then the end of e' that does not map to n must be in $B(b')$; call this vertex v. But, since $f(b') = b$, we must have that $f(v) \in B(b)$, and so e cannot be in $\mathrm{KILL}_b(H)$ and hence there can be no such n. So $U(\mathrm{KILL}_b(f))$ is a local isomorphism when $U(f)$ is.

Now we just have to deal with $\mathrm{COPY}_b(f)$. For convenience, we will call this f' for the remainder of the proof.

Suppose b is not in the image of f, and $U(f')$ is not a local isomorphism. So there is a node-vertex n in $U(G)$ such that $U(f')(n)$ is incident to an edge e not in the image of $U(f')$. e must be in the image of at least one of the p_i^H. Call (one of) its preimage(s) e'. Then e' must not be in the image of $U(f)$ (since otherwise e would be in the image of $U(f')$). But e' is incident to $U(f)(n)$, and so $U(f)$ is not a local isomorphism. Contrapositively, if $U(f)$ is is a local isomorphism, then so is $U(f')$.

Now suppose b is in the image of f. Let n be a vertex in $\mathrm{COPY}_{b'}(G)$ with an edge e incident to $f'(n)$ in $U(\mathrm{COPY}_b(H))$. e is in the image of one of the p_i^H; call the preimage of e under it e'. $f'(n)$ must also be in the image of p_i^H, and its preimage must be in the image of f. What is more, the preimage of this under f, which we will call n', must map to n by p_i^G.



We know that, since $U(f)$ is a local isomorphism, all the incident edges of $U(f)(n')$ must be in the image of $U(f)$, including e'. So e is in the image of $p_i^H \circ f$, and hence in the image of $f' \circ p_i^G$. So e is in the image of f', and hence $U(f')$ is a local isomorphism. □

We know that the two squares of (5.13) commute (up to an isomorphism) by lemma 5.1.5. For (5.17), we need the following lemma:

Lemma 5.4.3. *If $f : G_1 \to G_2$, $g : G_2 \to G_3$ and $h : G_3 \to G_4$ are !-graph monophisms that reflect !-box containment, and b is a !-vertex of G_3 not in the image of g then*

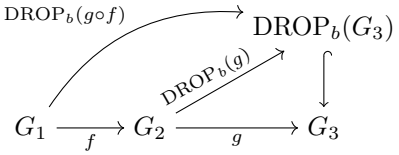
$$\mathrm{DROP}_b(g \circ f) = \mathrm{DROP}_b(g) \circ f$$

and

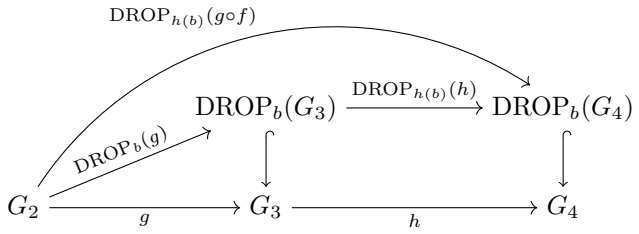
$$\mathrm{DROP}_{h(b)}(h \circ g) = \mathrm{DROP}_{h(b)}(h) \circ \mathrm{DROP}_b(g)$$

and similarly for KILL *and* COPY.

Proof. We can deduce the first DROP case from the following diagram, where the right and outer triangles commute by definition, and hence the left triangle commutes, since the inclusion map on the right is monic.



The second drop case follows in the same way from



and the KILL cases use the same argument.

For COPY, we recall that $\mathrm{COPY}_b(g \circ f)$ is defined to be $\iota_3 \circ \mathrm{KILL}_b(g \circ f)$, where $\iota_3 : \mathrm{KILL}_b(G_3) \to \mathrm{COPY}_b(G_3)$ is the inclusion map induced by the pushout square that defined $\mathrm{COPY}_b(G_3)$. We know that this is then $\iota_3 \circ \mathrm{KILL}_b(g) \circ f$, which is $\mathrm{COPY}_b(g) \circ f$ as required.

From the proof of 5.1.3, we note that

$$\mathrm{COPY}_{h(b)}(h) \circ \iota_3 = \iota_4 \circ \mathrm{KILL}_{h(b)}(h)$$

where $\iota_4 : \mathrm{KILL}_b(G_4) \to \mathrm{COPY}_b(G_4)$ is defined similarly to ι_3. Then we have

$$\begin{aligned}
\mathrm{COPY}_{h(b)}(h \circ g) &= \iota_4 \circ \mathrm{KILL}_{h(b)}(h \circ g) \\
&= \iota_4 \circ \mathrm{KILL}_{h(b)}(h) \circ \mathrm{KILL}_b(g) \\
&= \mathrm{COPY}_{h(b)}(h) \circ \iota_3 \circ \mathrm{KILL}_b(g) \\
&= \mathrm{COPY}_{h(b)}(h) \circ \mathrm{COPY}_b(g)
\end{aligned}$$

□

From this, we can deduce that the two squares of (5.17) commute. All we need now is that both squares in each diagram are pushouts. For this, we recall that pushouts of graphs are just unions of graphs. So a commuting square of monomorphisms

$$\begin{array}{ccc}
A & \xrightarrow{f} & B \\
{\scriptstyle g}\downarrow & & \downarrow{\scriptstyle h} \\
C & \xrightarrow[i]{} & D
\end{array}$$

is a pushout if and only if h and i cover D and the image of $h \circ f = i \circ g$ is exactly the intersection of the images of h and i. To get that this is the case for (5.17), we use the following lemma.

Lemma 5.4.4. *Let $f : G \to H$ be a !-graph monomorphism that reflects !-box containment, and let $b \in !(H)$. If $\iota_D : \mathrm{DROP}_b(H) \to H$ is the normal inclusion, then for any vertex or edge x of $\mathrm{DROP}_b(H)$, $\iota_D(x)$ is in the image of f if and only x is in the image of $\mathrm{DROP}_b(f)$, and similarly for KILL_b, and if $p_i^H : H \to \mathrm{COPY}_b(H)$ is one of the pushout maps from the definition of $\mathrm{COPY}_b(H)$, for any vertex or edge x of H, $p_i^H(x)$ is in the image of $\mathrm{COPY}_b(H)$ if and only if x is in the image of f.*

Proof. If b is not in the image of f, the arguments for all the operations are very similar; we will use DROP as the example. We know that the following diagram commutes (from lemma 5.4.1):

$$\begin{array}{ccc}
G & \xrightarrow{\mathrm{DROP}_b(f)} & \mathrm{DROP}_b(H) \\
& {\scriptstyle f}\searrow & \downarrow{\scriptstyle \iota} \\
& & H
\end{array}$$

Let x be a vertex or edge of $\mathrm{DROP}_b(H)$. If x is in the image of $\mathrm{DROP}_b(f)$, $\iota(x)$ must be in the image of f by commutativity of the diagram. Conversely, if $\iota(x)$ is in the image of f, its preimage must map to x under $\mathrm{DROP}_b(H)$, since ι is monic (and hence injective) and the diagram commutes.

So suppose b is in the image of f, and let b' be its preimage. We will start with the DROP case again. We have the following diagram (from lemma 5.1.3):

$$\begin{array}{ccc} \mathrm{DROP}_{b'}(G) & \xrightarrow{\mathrm{DROP}_b(f)} & \mathrm{DROP}_b(H) \\ {\scriptstyle \iota_G}\downarrow & & \downarrow{\scriptstyle \iota_H} \\ G & \xrightarrow{\quad f \quad} & H \end{array}$$

If x is a vertex or edge of $\mathrm{DROP}_b(H)$ that is in the image of $\mathrm{DROP}_b(f)$, then $\iota_H(x)$ is in the image of f, by commutativity of the diagram. For the converse, we will consider vertices and edges separately.

Suppose v is a vertex of $\mathrm{DROP}_b(H)$ with $\iota_H(v)$ in the image of f. Since f is monic, there is a unique v' with $f(v') = \iota_H(v)$. Now $\iota_H(v)$ cannot be b, so v' cannot be b', and hence v' is in the image of ι_G. But then the preimage of v' under ι_G must map to v by $\mathrm{DROP}_b(H)$, since all the maps are monic and the diagram commutes, so v is in the image of $\mathrm{DROP}_b(H)$.

Now suppose e is an edge of $\mathrm{DROP}_b(H)$ with $\iota_H(e)$ in the image of f. Again, there is a unique preimage e' under f, and the same must be true of the endpoints of the edge. So, by what we have already proved, the endpoints of e' must be in the image of ι_G, and hence e' must be (since ι_G is full in G), and we have what we require by commutativity of the diagram.

The KILL case is almost identical, except that in the vertex case, we must note that v' is not in $B(b')$, as v is not in $B(b)$.

The COPY case is also similar. This time we have the following diagram (actually two diagrams, with $i \in \{1, 2\}$).

$$\begin{array}{ccc} \mathrm{COPY}_{b'}(G) & \xrightarrow{\mathrm{COPY}_b(f)} & \mathrm{COPY}_b(H) \\ {\scriptstyle p_i^G}\uparrow & & \uparrow{\scriptstyle p_i^H} \\ G & \xrightarrow{\quad f \quad} & H \end{array}$$

As in the DROP case, if x is a vertex or edge of H that is in the image of f, then $p_i^H(x)$ is in the image of $\mathrm{COPY}_b(f)$.

Suppose v is a vertex of H with $p_1^H(v)$ in the image of $\mathrm{COPY}_b(f)$, which we will denote f' (the case for p_2^H is similar). We then have a vertex v' of $\mathrm{COPY}_{b'}(G)$ that maps to $p_1^H(v)$ by f'. Now v' is in the image of at least one of p_1^G and p_2^G. If v' is in the image of p_2^G, consider its preimage u in G. $f'(p_2^G(u)) = p_2^H(f(u)) = p_1^H(v)$, and hence v and $f(u)$ are both in the image of the inclusion $\iota_H : H \setminus B(b) \to H$, by the pushout that defines $\mathrm{COPY}_b(H)$, with a common preimage. But this means they must, in fact, be the same vertex, and hence v is in the image of f. The only other option is that v' is in the image of p_1^G; we will again call the preimage u. But then we have $f(u) = v$ by the above diagram of monomorphisms, and so v is in the image of f.

The argument for edges is similar to the DROP case. □

Theorem 5.4.5. *Consider the following rewrite, where m is a !-graph matching,*

$$\begin{array}{ccccc} L & \xleftarrow{i_1} & I & \xrightarrow{i_2} & R \\ {\scriptstyle m}\downarrow & & \downarrow{\scriptstyle d} & & \downarrow{\scriptstyle r} \\ G & \xleftarrow[g]{} & D & \xrightarrow[h]{} & H \end{array}$$

and let b be a !-vertex of G (we will also consider it to be a !-vertex of D and H). Then for any !-box operation OP_b on b, if b is not in the image of m, the following is a rewrite

$$\begin{array}{ccccc} L & \xleftarrow{i_1} & I & \xrightarrow{i_2} & R \\ {\scriptstyle \mathrm{OP}_b(m)}\downarrow & & \downarrow{\scriptstyle \mathrm{OP}_b(d)} & & \downarrow{\scriptstyle \mathrm{OP}_b(r)} \\ \mathrm{OP}_b(G) & \xleftarrow[\mathrm{OP}_b(g)]{} & \mathrm{OP}_b(D) & \xrightarrow[\mathrm{OP}_b(h)]{} & \mathrm{OP}_b(H) \end{array}$$

and if b is in the image of m, we consider it to be a !-vertex of L, I and R, and the following is a rewrite:

$$\begin{array}{ccccc} \mathrm{OP}_b(L) & \xleftarrow{\mathrm{OP}_b(i_1)} & \mathrm{OP}_b(I) & \xrightarrow{\mathrm{OP}_b(i_2)} & \mathrm{OP}_b(R) \\ {\scriptstyle \mathrm{OP}_b(m)}\downarrow & & \downarrow{\scriptstyle \mathrm{OP}_b(d)} & & \downarrow{\scriptstyle \mathrm{OP}_b(r)} \\ \mathrm{OP}_b(G) & \xleftarrow[\mathrm{OP}_b(g)]{} & \mathrm{OP}_b(D) & \xrightarrow[\mathrm{OP}_b(h)]{} & \mathrm{OP}_b(H) \end{array}$$

Proof. We have already established that both of these are commuting diagrams of !-graph monomorphisms, that the top span in each diagram is a !-graph rewrite rule and that $\mathrm{OP}_b(m)$ is a !-graph matching. All we have left to show is that all the squares are pushouts.

We will first consider the DROP_b case. Let $\iota_G : \mathrm{DROP}_b(G) \to G$ be the inclusion map, and similarly for the other !-graphs. Suppose we have a vertex or edge x in $\mathrm{DROP}_b(G)$. Then $\iota_G(x)$ must be in the image of at least one of m or g, and so x must (by lemma 5.4.4) be in the image of at least one of $\mathrm{DROP}_b(m)$ or $\mathrm{DROP}_b(g)$. So these maps cover $\mathrm{DROP}_b(G)$. If x is in the image of both these maps, $\iota_G(x)$ must be in the image of both m and g. Let x' be the preimage of x under $\mathrm{DROP}_b(g)$. Then $\iota_D(x')$ is the preimage of $\iota_G(x)$ under g, which is in the image of d, since this is a pushout of graphs. But then applying lemma 5.4.4 again gives us that x' must be in the image of $\mathrm{DROP}_b(d)$. Then commutativity of the diagrams and the fact that all the maps are monic gives us that the preimage of x under $\mathrm{DROP}_b(m)$ must be in the image of i_1 or $\mathrm{DROP}_b(i_1)$ (depending on whether b is in the image of m). So $\mathrm{DROP}_b(m)$ and $\mathrm{DROP}_b(g)$ intersect exactly on the part of the graph mapped to from I (or $\mathrm{DROP}_b(I)$), and so the left part of the diagram is a pushout. The same argument applies to the right squares, and so we have the result for DROP_b. KILL_b follows in the same way.

The argument for COPY_b is similar; here we have to note that if we have x in $\mathrm{COPY}_b(G)$, it must be in the image of at least one of the pushout maps from G that define $\mathrm{COPY}_b(G)$. We can call this map p_1^G and its preimage x'. Then, as before, we use lemma 5.4.4 to get that x is the in

the image of at least one of $\mathrm{COPY}_b(m)$ or $\mathrm{COPY}_b(g)$, since x' is in the image of at least one of m or g. Similarly, if x is in the image of both maps, it must be the image of something in $\mathrm{COPY}_b(I)$ or I (depending on whether b is in the image of m) since then x' is in the image of both m and g and is hence the image of something in I. □

In the above theorem, let $G \twoheadrightarrow H$ be the induced !-graph rewrite of the span $G \leftarrow D \rightarrow H$ in the rewrite. It should be clear that the rewrite rule induced by the span

$$\mathrm{OP}_b(G) \xleftarrow{\mathrm{OP}_b(g)} \mathrm{OP}_b(D) \xrightarrow{\mathrm{OP}_b(h)} \mathrm{OP}_b(H)$$

is exactly the result of applying OP_b to $G \twoheadrightarrow H$. So we have that if $G \twoheadrightarrow H$ is a !-graph rewrite derived from a rewrite $L \twoheadrightarrow R$, $\mathrm{OP}_b(G \twoheadrightarrow H)$ is derived from either $L \twoheadrightarrow R$ or $\mathrm{OP}_b(L \twoheadrightarrow R)$, and hence have the following corollary:

Corollary 5.4.6. *Suppose $L \twoheadrightarrow R$ is a !-graph rewrite rule that rewrites G to H at a !-graph matching, and $G \twoheadrightarrow H$ is the associated !-graph rewrite rule. Then for any concrete instance $G' \twoheadrightarrow H'$ of this rule, there is a concrete instance of $L \twoheadrightarrow R$ that rewrites G' to H', and the associated string graph rewrite rule $G' \twoheadrightarrow H'$ is a concrete instance of $G \twoheadrightarrow H$.*

This is the semantics of rewriting !-graphs using !-graph rewrite rules: that the rewrite holds for all concrete instances of the rewritten graphs. In this way, just as !-graph rewrite rules allowed us to represent inifinitely many string graph rewrite rules at once, !-graph rewriting allows us to do infinitely many string graph rewrites at once.

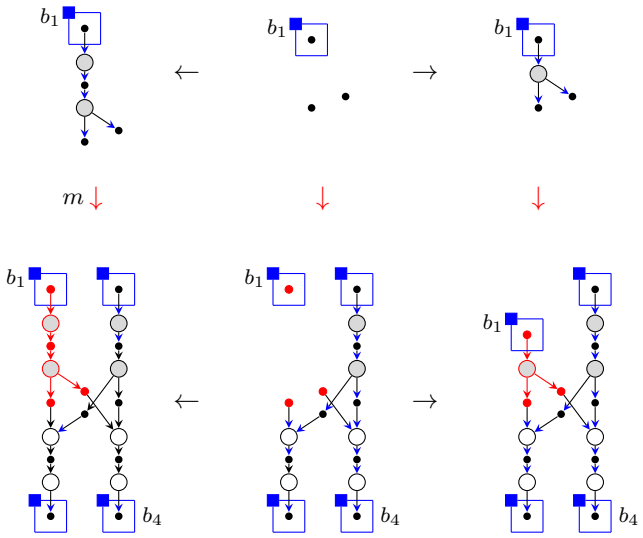
It is important to note that if we can apply an instantiation to a rewrite, we can apply it to a sequence of rewrites. If we consider two consecutive rewrites

$$G \twoheadrightarrow H \twoheadrightarrow J$$

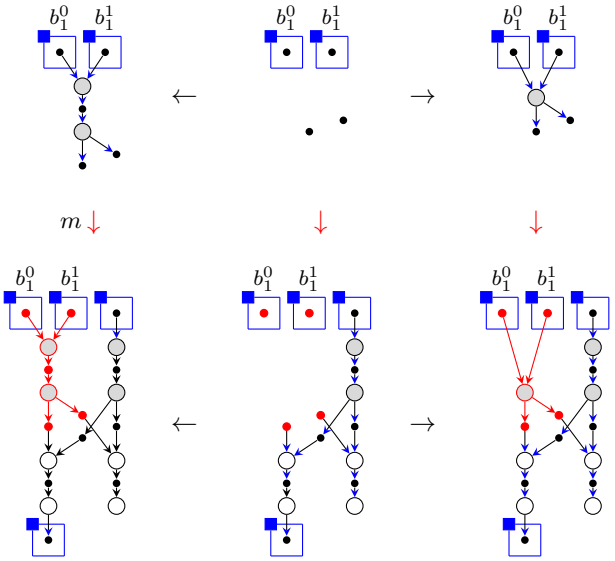
then we know that G and H contain the same !-vertices, and the same goes for H and J. If we can instantiate $G \twoheadrightarrow H$ to $G' \twoheadrightarrow H'$, applying the same instantiation to $H \twoheadrightarrow J$ will produce a rewrite from H' to some graph J', so we will have a rewrite sequence

$$G' \twoheadrightarrow H' \twoheadrightarrow J'$$

Example 5.4.7. Recall the rewrite in example 5.3.4:



If we perform COPY_{b_1} and KILL_{b_4}, we get the following rewrite:



Chapter 6

A Logic of !-Graphs

In this chapter, we will use the results of chapter 5 to build a logic of !-graphs that can be implemented by a computer, using graph rewriting as the core operation. This will include an equational logic much like that in section 3.5, together with some inference rules that build on that foundation.

In the term world, these rules would be based on predicates involving logical operators rather than purely on equations (although an equation is, of course, a type of predicate). For example, a $\bigwedge$-introduction rule might state that if you can provide a proof of a predicate P and a proof of a predicate Q, you can deduce that there is a proof of the predicate $P \wedge Q$. This would be written

$$\frac{P \qquad Q}{P \wedge Q}$$

Another common rule is induction, where P is a predicate over the natural numbers (or some other well-founded set):

$$\frac{P(0) \qquad P(n) \Rightarrow P(n+1)}{\forall n.P(n)}$$

Our graphical language does not contain logical connectives, nor does it contain the natural numbers used in the above induction rule. However, !-boxes do induce a well-founded structure amenable to a graphical analogue of induction. In this chapter, we will present this and other rules, and prove their soundness.

6.1 !-Graph Equational Logic

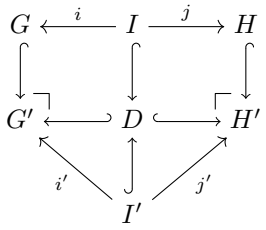
Let E be a set of !-graph equations. Our equational logic should look very familiar:

$$\text{(Axiom)}\ \frac{G \approx_{i,j} H \in E}{E \vdash G \approx_{i,j} H} \qquad \text{(Refl)}\ \frac{}{E \vdash G \approx_{b_G,b_G} G} \qquad \text{(Sym)}\ \frac{E \vdash G \approx_{i,j} H}{E \vdash H \approx_{j,i} G}$$

$$\text{(Trans)}\ \frac{E \vdash G \approx_{i,j} H \qquad E \vdash H \approx_{k,l} K}{E \vdash G \approx_{p,q} K} \qquad \text{(Leibniz)}\ \frac{E \vdash G \approx_{i,j} H}{E \vdash G' \approx_{i',j'} H'}$$

$$\text{(Homeo)}\ \frac{E \vdash G \approx_{i,j} H}{E \vdash G^* \approx_{i^*,j^*} H^*}$$

where $p := i$ and $q := l \circ k^{-1} \circ j$,



is a !-graph rewrite (where, in particular, all morphisms reflect !-box containment), and

$$G \xleftarrow{i} I \xrightarrow{j} H \qquad \text{and} \qquad G^* \xleftarrow{i^*} I \xrightarrow{j^*} H^*$$

are wire-homeomorphic.

To these, we add inference rules for instantiations of !-graph equations:

$$(\textsc{Copy})\ \frac{E \vdash G \approx_{i,j} H}{E \vdash \mathrm{COPY}_b(G \approx_{i,j} H)} \qquad (\textsc{Drop})\ \frac{E \vdash G \approx_{i,j} H}{E \vdash \mathrm{DROP}_b(G \approx_{i,j} H)}$$

$$(\textsc{Kill})\ \frac{E \vdash G \approx_{i,j} H}{E \vdash \mathrm{KILL}_b(G \approx_{i,j} H)}$$

Since our rewriting process implements Axiom and Leibniz up to wire homeomorphism and up to instantiation of !-graph rewrite rules, the above logic is equivalent to the relation $\overset{*}{\leftrightarrow}_E$.

For a set of !-graph equations E, let $\widetilde{E}$ be the set of concrete instances of the equations of E. Then the semantics of $E \vdash G \approx H$ we use is that, for any concrete instance $G' \approx H'$ of $G \approx H$,

$$\widetilde{E} \vdash G' \approx H'$$

in the equational logic of string graphs. Soundness of the above logic with respect to these semantics follows from corollary 5.4.6 and the fact that string graph rewriting implements the equational logic for string graphs.

6.2 Regular Forms of Instantiations

A lot of the proofs in this chapter rely on reasoning about instantiations. However, multiple instantiations can produce the same instance of a !-graph, making them harder than necessary to reason about. In particular, copying a !-box and then killing one of the copies results in the original graph (up to isomorphism).

In this section, we will demonstrate that it is possible to, if not completely remove these redunancies from the set of allowed instantiations, at least require a certain structure that is easy to reason about without affecting the set of possible instances of a !-graph.

We will concentrate on instantiations of !-graphs, but the results extend to !-graph equations.

6.2.1 Depth-Ordered Form

Definition 6.2.1 (Depth)**.** Let b be a !-vertex of a !-graph, and $P(b) = \text{pred}(b) \setminus b$ the set of parent !-vertices of b. Then the *depth* of b is defined by

$$\delta(b) = \begin{cases} 0 & P(b) = \varnothing \\ 1 + \max\{\delta(c) | c \in P(b)\} & \text{otherwise} \end{cases}$$

The depth of a !-box operation is the depth of the !-vertex it operates on.

Another way of viewing the depth of a !-vertex is that it is the longest ancestor-path to a top-level !-vertex.

We can now define a *depth-ordered form* for instantiations.

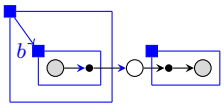
Definition 6.2.2 (Depth-ordered form)**.** A instantiation is *depth-ordered* if no !-box operation of depth n is preceded by an operation of depth greater than n.

So a depth-ordered instantiation has some number of !-box operations of depth 0, followed by some number of depth 1, then of depth 2 and so on.

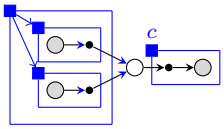
We will demonstrate that any instantiation has a depth-ordered form that is equivalent in the sense that the resulting graphs are isomorphic. To do this, we will present an algorithm that converts any instantiation into a depth-ordered one, where each change that the algorithm makes preserves the resulting graph.

We will need some results about what changes we can make, but first we need a way to talk coherently about those changes. For example, most of the changes we make will be commuting two operations in an instantiation.

An instantiation is a sequence of operations that progressively transforms a graph. The operations in that sequence cannot be divorced from the graphs they operate on. Consider an intermediate stage of a hypothetical instantiation; the operations up to this point have produced the graph

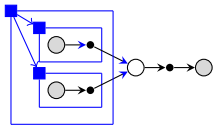
$G_1 =$ 

Suppose that the next operation in the sequence is COPY_b. Then the next intermediate graph will be

$G_2 =$ 

The depth of b in G_1, and hence of COPY_b, is 1. If the next operation is DROP_c, it will have depth 0, so the sequence cannot be depth-ordered.

What we would like to do is exchange the order of COPY_b and DROP_c, while still reaching the same graph G_3:

$G_3 =$ 

However, c is a vertex of G_2, and does not actually exist in G_1. c *is* derived from a !-vertex of G_1, though; which !-vertex is obvious from the pictures of the graphs.

More generally, we can construct a morphism from G_2 to G_1 that maps each vertex or edge to the one it was derived from.

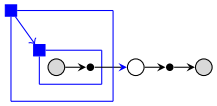
Definition 6.2.3 (Origin Map)**.** Let G_1 be a !-graph, $b \in !(G_1)$ and $G_2 = \text{COPY}_b(G_1)$. Then $f : G_2 \to G_1$, as defined by the following pushout:

$$\begin{array}{ccc} G_1 \setminus B(b) & \hookrightarrow & G_1 \\ \downarrow & & \downarrow \\ G_1 & \to & G_2 \\ & \searrow^{f} & \\ & & G_1 \end{array} \tag{6.1}$$

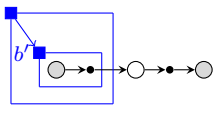
is called the *origin map* for COPY_b.

The origin maps for $\text{KILL}_b(G_1)$ and $\text{DROP}_b(G_1)$ are the natural inclusion morphisms into G_1 (which exist because these operations are graph subtractions).

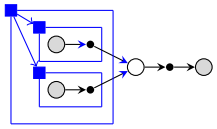
Armed with the origin map $f : G_2 \to G_1$, we can apply $\text{DROP}_{f(c)}$ to G_1, obtaining another graph G_2':

$G_2' =$ 

Now we need to find the !-vertex of G_2' that corresponds to b. In this case, since the origin map of a DROP operation is injective and b is in its image, it has a unique preimage b' in G_2'

$G_2' =$ 

and we can apply $\text{COPY}_{b'}$ to G_2' to get a graph G_3'

$G_3' =$ 

which is clearly isomorphic to G_3, and the origin maps for both sets of sequences are the same. COPY_b and DROP_c are an example of *commutable* operations.

Definition 6.2.4 (Commutable operations)**.** We say a sequence of two operations $\mathrm{OP}^1_b; \mathrm{OP}^2_c$ (with origin maps f_1 and f_2, respectively) is *commutable* when c is the only vertex that maps to $f_1(c)$ under f_1 and b has a unique preimage under f_2', the origin map of $\mathrm{OP}^2_{f_1(c)}$.

Note that whether two operations are commutable depends on the !-vertices they operate on. Of course, if we say that the operations are commutable, we expect them to have the same result in either order.

Proposition 6.2.5. *Let H_1 be a !-graph, $b \in !(H_1)$, $H_2 = \mathrm{OP}^1_b(H_1)$ with origin map f_1, $c \in !(H_2)$ and $H_3 = \mathrm{OP}^2_c(H_2)$ with origin map f_2. If OP^1_b and OP^2_c are commutable with commuted form $\mathrm{OP}^2_{c'}; \mathrm{OP}^1_{b'}$ (with origin maps f_2' and f_1' respectively) resulting in H_3', then $H_3 \cong H_3'$, and this commutes $f_1 \circ f_2$ and $f_2' \circ f_1'$.*

Proof. If both operations are DROP or KILL, we just note that the net effect is to remove the union of the subgraphs, regardless of the order. For example, if OP^1 is DROP and OP^2 is KILL,

$$H_3 = H_3' = H_1 \setminus (\{b\} \cup B(f_1(c)))$$

If OP^1 is COPY, the conditions on commutable operations mean that $f_1(c) \notin B(b)$. Suppose OP^2 is DROP. Consider the effect of $\mathrm{DROP}_{f_1(c)}; \mathrm{COPY}_b$ on H_1:

$$\begin{array}{ccc} (H_1 \setminus \{f_1(c)\}) \setminus B(b) & \longrightarrow & H_1 \setminus \{f_1(c)\} \\ \downarrow & & \downarrow \\ H_1 \setminus \{f_1(c)\} & \longrightarrow & H_3' \end{array}$$

Since $(H_1 \setminus \{f_1(c)\}) \setminus B(b) = (H_1 \setminus B(b)) \setminus \{f_1(c)\}$, H_3' must be the same as H_3, ie: $\mathrm{COPY}_b(H_1) \setminus \{\iota(f_1(c))\}$, where ι is the inclusion of $H_1 \setminus B(b)$ into $\mathrm{COPY}_b(H_1)$.

A similar argument works when OP^2 is KILL. We have

$$\begin{array}{ccc} (H_1 \setminus B(b)) \setminus B(f_1(c)) & \longrightarrow & H_1 \setminus B(f_1(c)) \\ \downarrow & & \downarrow \\ H_1 \setminus B(f_1(c)) & \longrightarrow & H_3' \end{array}$$

and, because every vertex in $B(\iota(f_1(c)))$ in $\mathrm{COPY}_b(H_1)$ must be in $B(f_1(c))$ in all copies of H_1 it is contained in, H_3' must again be the same as $\mathrm{COPY}_b(H_1) \setminus B(\iota(f_1(c)))$.

These arguments also work when OP^2 is COPY and OP^1 is DROP or KILL. That just leaves the case where both operations are COPY.

In their original order, we have two pushouts:

$$\begin{array}{ccc} H_1 \setminus B(b) & \longrightarrow & H_1 \\ \downarrow & & \downarrow{\scriptstyle p_1} \\ H_1 & \xrightarrow[p_2]{} & H_2 \end{array} \qquad \begin{array}{ccc} H_2 \setminus B(c) & \longrightarrow & H_2 \\ \downarrow & & \downarrow{\scriptstyle q_1} \\ H_2 & \xrightarrow[q_2]{} & H_3 \end{array}$$

Once commuted, we have

$$\begin{array}{ccc} H_1 \setminus B(f_1(c)) & \longrightarrow & H_1 \\ \downarrow & & \downarrow{\scriptstyle p'_1} \\ H_1 & \xrightarrow[p'_2]{} & H'_2 \end{array} \qquad \begin{array}{ccc} H'_2 \setminus B(b') & \longrightarrow & H'_2 \\ \downarrow & & \downarrow{\scriptstyle q'_1} \\ H'_2 & \xrightarrow[q'_2]{} & H'_3 \end{array}$$

Note that the requirements for commutability of operations mean that c is in the image of both p_1 and p_2, and b' is likewise in the image of both p'_1 and p'_2. We construct the isomorphism from H_3 to H'_3 by initially constructing a bijective map ϕ_v from the vertices of H_3 to the vertices of H'_3.

Let v_3 be a vertex of H_3, $v_2 = f_2(v_3)$ and $v_1 = f_1(v_2)$. We then choose v'_2 to be $p'_1(v_1)$ if v_3 is in the image of q_1 and $p'_2(v_1)$ if v_3 is in the image of q_2. Likewise, $v'_3 = q'_1(v'_2)$ if v_2 is in the image of p_1 and $v'_3 = q'_2(v'_2)$ if v_2 is in the image of p_2. We then set $\phi_v(v_3) = v'_3$.

First, we need to check that we have defined a coherent function. If v_3 is in the image of both q_1 and q_2, v_2 cannot be in $B(c)$. Then v_1 is not in $B(f_1(c))$, since if it were, whichever of p_1 and p_2 maps v_1 to v_2 must also map the edge from $f_1(c)$ to v_1 to an edge from c to v_2. Thus $p'_1(v_1) = p'_2(v_1)$. Likewise, if v_2 is in the image of both p_1 and p_2, v'_2 cannot be in $B(b')$ and $q'_1(v'_2) = q'_2(v'_2)$. So ϕ_v is a valid function.

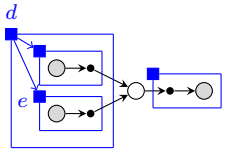
Note that the definition of an origin map means that $f'_1(v'_3) = v'_2$ and $f'_2(v'_2) = v_1$. Thus the same construction works in reverse, allowing us to construct the inverse map ϕ_v^{-1}, and so ϕ_v is a bijection.

Since !-graphs are simple, we just need to show that, for any two vertices v_3 and w_3 of H_3, there is an edge from v_3 to w_3 if and only if there is one from $\phi_v(v_3)$ to $\phi_v(w_3)$. We construct a w family of vertices in the same way we constructed the v family.

If there is an edge e_3 from v_3 to w_3, this edge is mapped to an edge e_1 from v_1 to w_1 by $f_1 \circ f_2$. The existence of e_3 means that v_3 and w_3 must both be in the image of q_1 or both be in the image of q_2. Thus v'_2 and w'_2 must both be in the image of p'_1 or both be in the image of p'_2, and that morphism must map e_1 to an edge e'_2 from v'_2 to w'_2. Similarly, either q'_1 or q'_2 must map e'_2 to an edge from v'_3 to w'_3.

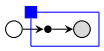
The converse case is symmetric, and hence ϕ_v extends to an isomorphism $\phi : H_3 \cong H'_3$. □

We have generalised commutable operations beyond just being on entirely separate parts of the graph, like in the example above, but we still have to deal with non-commutable operations. If the operations are already depth-ordered, such as if the next operation were DROP_e

$G_2 =$ 

we can safely ignore them.

Instead, we will look at operations where the preimage of the !-vertex the second operation acts on is a parent of the !-vertex the first acts on. For example, suppose the next operation on G_2 were KILL_d. Then the operations would not be depth ordered (KILL_d has depth 0, like DROP_c did, while COPY_b has depth 1), but they cannot be commuted; once again, b is not in the image of the origin map of $\mathrm{KILL}_{f(d)}$.



In this case, we can simply discard the COPY_c operation, since KILL_d is clearly isomorphic to $\mathrm{KILL}_{f(d)}$. The same would be true if the first operation were DROP_c or KILL_c.

Lemma 6.2.6. *Let H_1 be a !-graph, $b \in !(H_1)$, $H_2 = \mathrm{OP}_b(H_1)$ with origin map f, $c \in !(H_2)$ and $H_3 = \mathrm{KILL}_c(H_2)$ with origin map k, where $b \in B(f(c))$. Then H_3 is isomorphic to $\mathrm{KILL}_{f(c)}$ (with origin map k'), and this isomorphism commutes k' and $f \circ k$.*

Proof. If OP is DROP or KILL, this has the same net effect, since $b \in B(f(c))$ and hence $B(b) \subseteq B(f(c))$. If it is COPY, it is sufficient to note that

$$H_1 \setminus B(f(c)) \subseteq H_1 \setminus B(b)$$

and hence the COPY operation has no effect on the part of the graph preserved by the KILL operation. □

If the next operation is COPY_d, b will have *two* preimages under the origin map of $\mathrm{COPY}_{f(d)}$, which we will call b^0 and b^1. Then the replacement sequence will be

$$\mathrm{COPY}_{f(d)}; \mathrm{COPY}_{b^0}; \mathrm{COPY}_{b^1}$$

Lemma 6.2.7. *Let H_1 be a !-graph, $b \in !(H_1)$, $H_2 = \mathrm{OP}_b(H_1)$ with origin map f, $c \in !(H_2)$ and $H_3 = \mathrm{COPY}_c(H_2)$ with origin map g, where $b \in B(f(c))$.*

Let $H_2' = \mathrm{COPY}_{f(c)}(H_1)$, with origin map g'. Then b has two preimages under g', b^0 and b^1. Let $H_3' = \mathrm{OP}_{b^0}$, with origin map f_0'. b^1 has a unique preimage under f_0', which we call b'^1. Let $H_4' = \mathrm{OP}_{b'^1}$, with origin map f_1'. Then H_4' is isomorphic to H_3, and this isomorphism commutes $f \circ g$ and $g' \circ f_0' \circ f_1'$.

Proof. Intuitively, whatever we did to b before the COPY operation, we do instead to both copies of b after it.

If OP is DROP, we remove both b^0 and b^1, and this is the same as removing just b before applying the COPY. If OP is KILL, the preimage of $B(b)$ under g' is exactly the union of $B(b^0)$ and $B(b^1)$, and hence the net effect is likewise maintained. We note that $B(b^0) \cap B(b^1) = \varnothing$, and then the COPY case follows from a similar proof to that of two commutable COPY operations (in proposition 6.2.5). □

We have not dealt with the second operation being DROP_d, because this commutes with COPY_c, and indeed DROP commutes with any preceding operation on a distinct !-vertex, including child !-vertices.

When we are dealing with rewrite sequences longer than two operations, explicitly referring to the origin maps will become extremely cumbersome. Therefore, if we have an origin map $f : G_2 \to G_1$ and a !-vertex b of G_1 with a unique preimage under f, we will also refer to this preimage as b. If it has two preimages under f, we will refer to one as b^0 and the other as b^1.

We can now set out algorithm 1, which takes an arbitrary instantiation of a !-graph and gives a depth-ordered one that produces the same graph.

Algorithm 1 Order !-box operations by depth

while has unmarked operations **do**
  $op \leftarrow$ rightmost unmarked operation
  $b \leftarrow$!-box op operates on
  look at op', immediately to right of op, operating on b'
  if there is no such op' **then** {case 1}
    mark op
  else if $\delta(b) \leq \delta(b')$ **then** {case 2}
    mark op
  else if b' is not a parent of b **then** {case 3}
    commute op and op'
  else if op' is KILL **then** {case 4}
    discard op
  else if op' is DROP **then** {case 5}
    commute op and op'
  else if op' is COPY **then** {case 6}
    remove op and add two copies of op to the right of op', acting on the two copies of b
  end if
end while

The preceding discussion gives rise to the following proposition, stating that the algorithm does not alter the effect of the instantiation:

Proposition 6.2.8. *Applying algorithm 1 to an instantiation that results in a graph G yields (if it terminates) an instantiation that results in a graph isomorphic to G.*

The main invariants of the algorithm are that the marked operations are depth-ordered relative to each other, and that no unmarked operation of depth d is ever to the right of a marked operation of depth $> d$. This can be seen by noting that op is always unmarked and everything to the right of it, including op', is marked. The first part of the invariant is preserved because the algorithm never alters the order of the marked operations, and an operation is only marked when the marked operations to its right have the same or greater depth (and the second part of the invariant requires that the marked operations to the left have the same or lesser depth). The second part of the invariant is maintained because we only ever move op to the right of op' when $\delta(op) > \delta(op')$ and while this can reduce the depth of op, it can only reduce it to $\delta(op')$.

Proposition 6.2.9. *Algorithm 1 terminates.*

Proof. We will place a (tight) upper bound on the number of iterations, as well as on the length of the final instantiation. Specifically, given the recurrence relations

$$s_0 = 0$$
$$s_{i+1} = s_i + 2^{s_i}$$
$$c_0 = 0$$
$$c_{i+1} = c_i + 2^{s_i} - 1$$

we will show that if the input instantiation is of length i, the output instantiation is no longer than s_i operations and the algorithm takes no more than $t_i = c_i + s_i$ iterations. We will do this by induction on i. The $i = 0$ case is trivial.

The main thing to note for the step case is that if we have the instantiation $S = \mathrm{OP}_b; T$, because the algorithm always deals with the rightmost unmarked operation (and the operation to its right), it will not consider OP_b until it is the last remaining unmarked operation. This means that the first part of a run of the algorithm on S is the same as a run of the algorithm on T.

So suppose the length of S is $i + 1$. By the inductive hypothesis, after at most t_i iterations, the instantiation will look like $\mathrm{OP}_b; T'$, where T' consists of at most s_i marked (and no unmarked) operations, and OP_b remains unmarked.

Let l be the actual length of T'. We will show that after at most $2^l - 1$ iterations that hit cases 3-6, the instantiation will be depth-ordered. Since only case 6 iterations can introduce new operations, and only one at that, there can be at most 2^l case 1 or 2 iterations before the entire sequence is marked. So there are at most $l + 2^l \leq s_i + 2^{s_i}$ operations in the final sequence, and at most

$$t_i + 2^l + 2^l - 1 = s_i + 2^l + c_i + 2^l - 1 \leq s_{i+1} + c_{i+1} = t_{i+1}$$

iterations in total.

We need to note here that any operations in the sequence, marked or unmarked, at any subsequent point that are not part of T' must be OP_b or a copy of OP_b. If two copies of OP_b are adjacent, they must have the same depth. This is because the depth of a copy of OP_b can only be changed by being commuted with a DROP_c. But any other copy of OP_b adjacent to it must have been commuted with the same DROP_c, and its depth will have been changed in the same way. It follows that cases 3-6 can only be triggered if op' is a member of T'.

Suppose we are at some iteration of the algorithm (at or after the point where we have $\mathrm{OP}_b; T'$). Consider op, the rightmost unmarked operation in the sequence. Call the subsequence to the left of op S_1, and the subsequence to its right S_2, so the whole sequence is $S_1; op; S_2$. Let n be the number of operations of T' that are in S_2.

We will show by induction on n that, in no more than $2^n - 1$ case 3-6 iterations (and some number of case 1 or 2 iterations), the instantiation will be S_1 followed by marked operations.

For $n = 0$, either S_2 is empty, in which case we trigger case 1 and are done, or S_2 consists only of operations derived from OP_b, which must therefore have the same depth as *op*. This means we trigger case 2, and again are done.

Suppose $n = k + 1$. If cases 1, 2 or 4 are triggered, we are done. Cases 3 and 5 require the inductive hypothesis, but are straightforward. So suppose we hit case 6. We now get two copies of *op*, each of which has k operations from T' to the right of it. We apply the inductive hypothesis to the right hand one to get that in at most $2^k - 1$ case 3-6 iterations, the left hand one will be the rightmost unmarked operation. As already discussed, this will still have k operations from T' to the right of it, so after no more than $2^k - 1$ further case 3-6 iterations, we will have the sequence we want (S_1 followed by only marked operations). Then we see that

$$1 + (2^k - 1) + (2^k - 1) = 2^{k+1} - 1$$

which is what we wanted. □

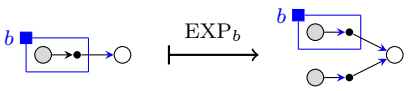
Corollary 6.2.10. *Any instance H of a !-graph G has a depth-ordered instantiation.*

The upper bound for this algorithm may appear worryingly high, but it is not an algorithm that would ever be implemented. The fact that such an algorithm exists, however, allows us to assume that any instantiation is depth-ordered; if it is not, we can produce an equivalent one that is. In fact, in chapter 7 we will demonstrate matching techniques that directly produce depth-ordered instantiations.

6.2.2 Expansion-Normal Form

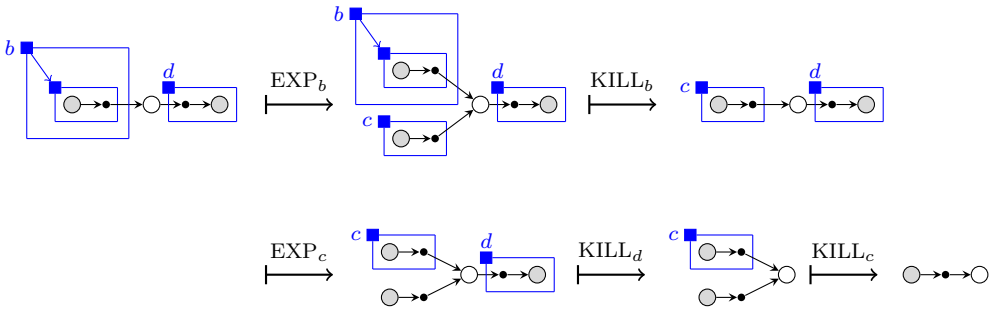
A depth-ordered concrete instantiation has only operations of depth 0. To see this, note that any operation of depth greater than 0 must be on a !-box that has one or more parents, and at least one of those parents must be of depth 0 (by posetality of the subgraph of !-vertices). However, since the instantiation is depth-ordered, there can be no further operations on that !-box, and so it must exist in the final graph, which cannot then be a concrete graph. So the instantiation cannot be concrete.

We will introduce EXP_b as a shorthand for $\mathrm{COPY}_b; \mathrm{DROP}_{b^1}$. As a notational convenience, we will consider b^0 to be the same as b, since it is the unique preimage of b under the composition of the origin maps.



Definition 6.2.11 (Expansion-normal form)**.** A concrete instantiation is in *expansion-normal form* when it is composed entirely of EXP and KILL operations, and the depth of every operation is 0.

Example 6.2.12. A sequence in expansion normal form:



Every operation is on a top-level !-box (depth 0), and the only operations are EXP and KILL.

We generally identify the "constant" parts of the graph for each operation (in a manner consistent with the origin maps), and view EXP operations as adding to the graph and KILL operations as removing part of the graph.

Theorem 6.2.13. *Any concrete instance H of a !-graph G has an instantiation in expansion-normal form.*

Proof. We know from corollary 6.2.10 that there must be a depth-ordered instantiation of H from G; we will transform this into an equivalent instantiation (ie: resulting in the same graph, up to isomorphism) that is in expansion-normal form.

In order to do this, we will need to be able to, for a given COPY_b operation, move any operation on b^0 or b^1 to immediately after COPY_b. We will demonstrate this for arbitrary depth-ordered instantiations, not just concrete ones.

Consider an operation OP_{b^0} (recalling that b^0 and b^1 are arbitrary labels, and hence interchangable). The depth of OP_{b^0} must be the same as the depth of COPY_b. The only way the depth of OP_{b^0} could deviate from that of COPY_b is if there were a DROP operation between them on a parent of b^0. But the depth of this DROP operation would have to be strictly less than the depth of COPY_b, which contradicts the depth-ordering. Indeed, there can be no operations on any parent of b^0 after COPY_b.

Similarly, there can be no operations on any child of b^0 between COPY_b and OP_{b^0}, since that operation would have a depth strictly greater than that of b^0, and hence of OP_{b^0}, which violates depth-ordering. Therefore, proposition 6.2.5 allows OP_{b^0} to be freely exchanged with the operation preceeding it until it reaches COPY_b, and this will maintain the depth-ordering of the sequence.

Note that this also allows us to move EXP_{b^0} back, since we can move COPY_{b^0} back to COPY_b, and then move $\mathrm{DROP}_{b^{01}}$ back to COPY_{b^0}. Likewise, given EXP_c, we can move any subsequent EXP_c back to join it, so we can move all occurrences of EXP_{b^0} and EXP_{b^1} to immediately follow COPY_b.

Starting with a depth-ordered concrete instantiation, we can eliminate COPY (in favour of EXP) with the following procudure (note that we treat EXP as an opaque operation in its own right, not a composite of COPY and DROP), and so obtain a concrete instantiation in expansion-minimal form.

We eliminate COPY operations starting from the right. At each stage, we consider the rightmost COPY operation, COPY_b. If the next operations on b^0 and b^1 are both EXP, we first move all the EXP_{b^0} and EXP_{b^1} operations to immediately after COPY_b. We then replace all the EXP_{b^1} operations by EXP_{b^0}, which is an equivalent operation in this context.

This done, we know that, after COPY_b, the first operation on b^1 must be either KILL_{b^1} or DROP_{b^1}. In both cases, we move it back so that it immediately follows COPY_b. In the case of DROP_{b^1}, we can just replace both operations with EXP_b (and replace all later references to b^0 with b). Since COPY_b was depth 0, EXP_b must also be depth 0. In the case of KILL_{b^1}, we can eliminate both operations (and, again, replace all later references to b^0 with b). Then we will have eliminated this occurrence of COPY, and we can proceed onto the next rightmost occurrence, until none remain.

Once we have eliminated all occurrences of COPY, we can eliminate DROP operations by replacing each DROP_b with $\mathrm{EXP}_b; \mathrm{KILL}_b$, which both have depth 0 since DROP_b does. $\mathrm{EXP}_b; \mathrm{KILL}_b$ is $\mathrm{COPY}_b; \mathrm{DROP}_{b^0}; \mathrm{KILL}_{b^1}$, and this is equivalent to $\mathrm{COPY}_b; \mathrm{KILL}_{b^1}; \mathrm{DROP}_{b^0}$, which is equivalent to DROP_b.

So we now have a concrete instantiation containing only EXP and KILL operations, and all the operations are of depth 0, so it is in expansion-normal form. □

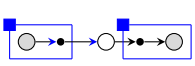
Once we have an instantiation in expansion-normal form, if b is a top-level !-vertex in the starting graph G, we can pull all the operations on b to the start of the instantiation. So the instantiation will start with some number of EXP_b operations, followed by KILL_b (the argument for this is identical to the one that allowed us to move operations on b^0 back to COPY_b).

6.3 !-box Introduction

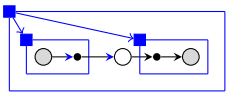
We will demonstrate a simple inference rule that allows us to wrap an entire !-graph equation in a !-box.

Definition 6.3.1 (BOX)**.** Suppose G is a !-graph. Then $\mathrm{BOX}(G)$ is the $\mathcal{G}_{T!}$-typed graph consisting of G together with a fresh !-vertex b and an edge from b to every vertex in the graph (including itself).

Example 6.3.2. If we consider the graph G:



then $\mathrm{BOX}(G)$ is



We will sometimes write $\mathrm{BOX}_b(G)$ to specify a name for the fresh !-vertex; in contrast to the other !-box operations, b is not in $!(G)$ but is in $!(\mathrm{BOX}_b(G))$.

$\mathrm{BOX}(G)$ is easily seen to be a !-graph: $U(\mathrm{BOX}(G)) \cong U(G)$ and hence is a string graph; extending a poset with a fresh element that is $\leq$ every element of the poset and to itself yields another poset; $U(B_{\mathrm{BOX}(G)}(c)) \cong U(B_G(c))$ for every $c \in !(G)$ and $U(B(b)) \cong U(G)$, so these are all trivially open subgraphs of $U(\mathrm{BOX}(G))$; and $c' \in B(c) \Rightarrow B(c') \subseteq B(c)$ follows from the fact this is true for G and $B(b) = \mathrm{BOX}(G)$.

As with the other operations, we can apply BOX to monomorphisms that reflect !-box containment:

Proposition 6.3.3. *Let $f : G \to H$ be a monomorphism in $\mathbf{BGraph}_T$ that reflects !-box containment. Then f can be uniquely extended to a !-box-reflecting monomorphism $\mathrm{BOX}(f) : \mathrm{BOX}(G) \to \mathrm{BOX}(H)$ that commutes with the inclusions of G into $\mathrm{BOX}(G)$ and H into $\mathrm{BOX}(H)$.*

Further, if $g : H \to K$ is another such morphism, $\mathrm{BOX}(g \circ f) = \mathrm{BOX}(g) \circ \mathrm{BOX}(f)$.

Proof. If we have $\mathrm{BOX}_b(G)$ and $\mathrm{BOX}_c(H)$, the only possible extension of f is the one that maps b to c, and maps each edge between b and some vertex v in $\mathrm{BOX}_b(G)$ to the edge from c to $f(v)$ in $\mathrm{BOX}_c(H)$. This then trivially reflects !-box containment and is monic, since f is. It also trivially respects composition. □

Applying BOX to every morphism in definition 5.1.2 gives us the following:

Corollary 6.3.4. *If $G \xleftarrow{i} I \xrightarrow{j} H$ is a !-graph equation, then so is*

$$\mathrm{BOX}(G) \xleftarrow{\mathrm{BOX}(i)} \mathrm{BOX}(I) \xrightarrow{\mathrm{BOX}(j)} \mathrm{BOX}(H)$$

.

We can now state the inference rule

$$\textsc{Box}\ \frac{E \vdash G \approx H}{E \vdash \mathrm{BOX}(G \approx H)}$$

Theorem 6.3.5. Box *is sound. In particular, if for all concrete instances $G' \approx H'$ of $G \approx H$,*

$$\widetilde{E} \vdash G' \approx H'$$

then for any concrete instance $G'' \approx H''$ of $\mathrm{BOX}_b(G \approx H)$,

$$\widetilde{E} \vdash G'' \approx H''$$

Proof. By theorem 6.2.13, there is an instantiation S of $G'' \approx H''$ from $\mathrm{BOX}_b(G \approx H)$ in expansion-normal form, and we can further require that S decomposes into $T; U$ where T is a sequence of EXP_b operations, followed by KILL_b.

Applying EXP_b to $\mathrm{BOX}_b(G)$ produces the disjoint union of $\mathrm{BOX}_b(G)$ and G. The next EXP_b adds another copy of G, and so on. Finally, KILL_b removes the copy of $\mathrm{BOX}_b(G)$, leaving zero or more disjoint copies of G. So applying T to $G \approx H$ will produce $G_n \approx H_n$, where G_n is n disjoint copies of G and H_n is n disjoint copies of H.

Now if we apply U to $G_n \approx H_n$, we know that each operation can only affect a single copy of G in G_n and the corresponding copy of H in H_n. There are thus instantiations $U_1, \ldots, U_n$ of $G \approx H$ such that $U(G_n)$ is $U_1(G) \uplus \cdots \uplus U_n(G)$ and similarly for $U(H_n)$ (where we are using the notation from remark 4.4.2).

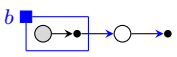
We can then use the assumption to construct proofs of $\widetilde{E} \vdash U_i(G \approx H)$ for each $1 \leq i \leq n$, and use Leibniz to apply these to the decomposed parts of $U(G_n \approx H_n)$. □

This is mostly useful in the construction of larger proofs. We will demonstrate its use in section 6.4.1.

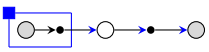
6.4 !-box Induction

In this section, we introduce an analogue of induction for !-graphs. This will follow the usual induction scheme of having a base case and a step case that must be proved for a particular !-vertex b. However, in the absence of the named variables of terms, we need some way to link a !-vertex in the inductive hypothesis to one in the equation we need to prove for the step case.

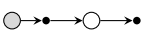
We will introduce the possibility of *fixing* a !-vertex, which prevents any !-box operations being performed on it during the matching process. The idea is that if we take the following graph



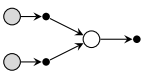
and fix b, it should match itself and



but not

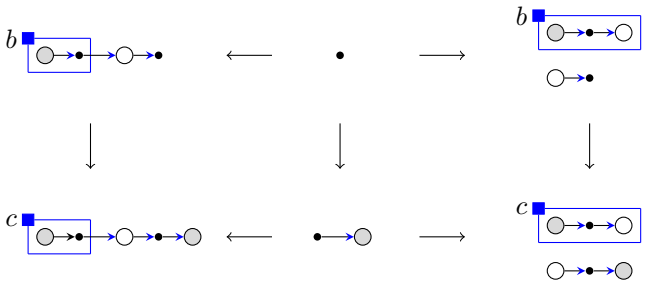


or



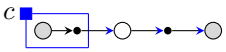
as it normally would. The result of this is that, given a rule like

where b is fixed, the only possible rewrites are ones like



where the !-box operations that can subsequently be performed on b and c (and hence the possible concrete instantiations) are linked.

In order to deal with multiple applications of the induction rule, we will tag these fixed !-vertices with arbitrary symbols drawn from a (countable) set $\mathcal{A}$. So we could tag b above with $x \in \mathcal{A}$; in this case, we say that b is *x-fixed*, and we only allow b to match c in



if c is also x-fixed.

To store this information, we augment a !-graph G with a partial map

$$\mathrm{fix}_G :!(G) \rightharpoonup \mathcal{A}$$

that maps each fixed !-vertex to its tag. We require !-graph morphisms to preserve these tags, and the morphisms of a !-graph equation to also reflect them. In particular, if we have the !-graph equation $G \approx H$, the fact we can consider the !-vertices of G and H to be the same means we can view fix_G as being the same as fix_H, and call these $\mathrm{fix}_{G \approx H}$.

This will have an impact on the !-graph logic, and the notion of soundness we are considering. Specifically, we will consider what we mean for $E \vdash G \approx H$ to hold. Let $A = \mathrm{im}(\mathrm{fix}_{G \approx H})$. For each $x \in A$, each non-negative integer $n \in \mathbb{N}_0$, and each !-graph equation $L \approx R$, we define $X^x_n(L \approx R)$ to be $L \approx R$ with n EXP_b operations and one KILL_b operation applied to it for each x-fixed !-vertex b in the equation. Then for each function $\lambda \in \mathbb{N}_0^A$ from A to the non-negative integers, we define $X_\lambda(L \approx R)$ to be $L \approx R$ with $X^x_{\lambda(x)}$ applied to it for each $x \in A$ (the order is irrelevant, as these are all top-level !-vertices). Then the semantics of $E \vdash G \approx H$ is that for each $\lambda \in \mathbb{N}_0^A$ and each instance $G' \approx H'$ of $X_\lambda(G \approx H)$, if we let

$$E_\lambda = \{X_\lambda(L \approx R) | L \approx R \in E . L \approx R \text{ has no } x\text{-fixed !-vertices for } x \notin A\}$$

then

$$\widetilde{E_\lambda} \vdash G' \approx H'$$

in the equational logic of string graphs.

We need to adjust the !-graph logic and rewriting to keep it sound under these modified semantics. We place a side constraint on COPY, DROP and KILL that the !-vertex they operate on cannot be fixed (hence the term "fixed"), and on BOX that the equation contains no fixed !-vertices. Likewise, when rewriting, we do not allow any !-box operations to be applied to fixed !-vertices.

Our previous requirement that morphisms preserve fixing tags means that the matching morphisms of a !-graph rewrite can only map x-fixed !-vertices to other x-fixed !-vertices (although an unfixed !-vertex may match an x-fixed one). This also constrains the LEIBNIZ rule.

We also introduce a !-box operation $\mathrm{FIX}_b^x(G)$ that produces a !-graph identical to G but where the !-vertex b is x-fixed (and similarly for !-graph equations), and a rule

$$(\textsc{Fix})\ \frac{E \vdash G \approx_{i,j} H}{E \vdash \mathrm{FIX}_b^x(G \approx_{i,j} H)}$$

with the condition that b is a top-level !-vertex of $G \approx_{i,j} H$ that is not fixed.

In the pictoral representations of !-graphs, we will mark x-fixed !-vertices with the label F_x.

The following is the inference rule for !-box induction, based on one originally suggested as a possibility by Kissinger in his thesis ([23], pp 179-181). It holds providing b is a top-level, unfixed !-vertex in $G \approx H$ and x is a fresh tag (that does not already occur in $G \approx H$ or any of the equations in E).

$$(\textsc{Induct})\ \frac{E \vdash \mathrm{KILL}_b(G \approx H) \qquad E \cup \{\mathrm{FIX}_b^x(G \approx H)\} \vdash \mathrm{FIX}_b^x(\mathrm{EXP}_b(G \approx H))}{E \vdash G \approx H}$$

Theorem 6.4.1. INDUCT *is sound.*

Proof. Let $A = \mathrm{im}(\mathrm{fix}_{G \approx H})$. We need to show that for each $\lambda \in \mathbb{N}_0^A$ and each concrete instance $G' \approx H'$ of $X_\lambda(G \approx H)$,

$$\widetilde{E_\lambda} \vdash G' \approx H'$$

in the equational theory of string graphs.

We can choose an instantiation of $G' \approx H'$ in expansion-normal form with all the operations on b at the start (since b is top-level). Denote this $T; U$, where T contains all the operations on b, and U all the other operations. T is comprised of n expansions, for some n, followed by KILL_b.

We show by induction on n that, for all concrete instances $G_V \approx H_V$ of $X_\lambda(G \approx H)$ with an instantiation of the form $T; V$,

$$\widetilde{E_\lambda} \vdash G_V \approx H_V$$

$n = 0$: T is just KILL_b. It follows from the definition of X_λ that there is an instantiation W of $X_\lambda(G \approx H)$ from $G \approx H$ such that W only operates on top-level !-vertices of $G \approx H$. Then $W; T; V$ is equivalent to $T; W; V$, and so $G_V \approx H_V$ is an instance of $\mathrm{KILL}_b(G \approx H)$. The assumption $E \vdash \mathrm{KILL}_b(G \approx H)$ then gives us that

$$\widetilde{E_\lambda} \vdash G_V \approx H_V$$

$n = k + 1$: let T' be T with the first expansion removed (so $T = \mathrm{EXP}_b; T'$). We know that T' is equivalent to X_k^x, and so V is an instantiation of $G_V \approx H_V$ from $X_k^x(\mathrm{EXP}_b(X_\lambda(G \approx H)))$. But

$$X_k^x(\mathrm{EXP}_b(X_\lambda(G \approx H))) = X_\lambda(X_k^x(\mathrm{EXP}_b(G \approx H)))$$

since all the operations are on top-level !-vertices of $G \approx H$. Further, we can define

$$\mu(y) = \begin{cases} \lambda(y) & y \in A \\ k & x = y \end{cases}$$

and then

$$X_\lambda(X_k^x(\mathrm{EXP}_b(G \approx H))) = X_\mu(\mathrm{FIX}_b(\mathrm{EXP}_b(G \approx H)))$$

So now we have that V is an instantiation of $G_V \approx H_V$ from $X_\mu(\mathrm{FIX}_b(\mathrm{EXP}_b(G \approx H)))$, and then the step case assumption means we can construct a proof of

$$\widetilde{E_\mu} \vdash G_V \approx H_V$$

Now the way μ was defined means that

$$E_\mu = E_\lambda \cup \{X_\lambda(X_k^x(G \approx H))\}$$

and so the proof may contain invocations of Axiom with a concrete instance of $X_\lambda(X_k^x(G \approx H))$. Consider one such invocation, where the axiom $G_W \approx H_W$ is a (concrete) instance of $X_\lambda(X_k^x(G \approx H))$, which is the same as $X_k^x(X_\lambda(G \approx H))$. Then $G_W \approx H_W$ is an instance of $X_\lambda(G \approx H)$ and there is an instantiation of the form $T'; W$ witnessing this. The inductive hypothesis then allows us to construct a proof of

$$\widetilde{E_\lambda} \vdash G_W \approx H_W$$

We can thus replace all such Axiom invocations with proof trees using only the axioms of E_λ, and so discard $X_\lambda(X_k^x(G \approx H))$ from E_μ, leaving us with a proof of

$$\widetilde{E_\lambda} \vdash G_V \approx H_V$$

as required, and the induction is complete.

Now we just let $V = U$, and we have

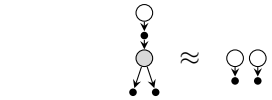
$$\widetilde{E_\lambda} \vdash G' \approx H'$$

as required. □

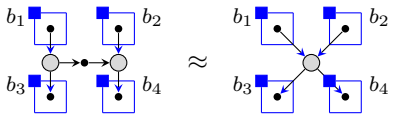
This can be viewed as a sort of !-box introduction rule, more powerful than Box (although relying on correspondingly stronger assumptions). It can be used to produce !-versions of concrete equations, such as those generated by an automated tool like QuantoCosy[23].

Example 6.4.2. Suppose we have the following three graph equations, drawn from the Z/X calculus presented in section 2.2:

(6.2)

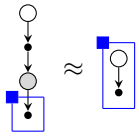


(6.3)



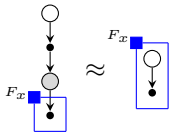
(6.4)

The first two are easily found by QuantoCosy, and the latter is the spider theorem arising from commutative Frobenius algebras. We wish to use !-box induction to show

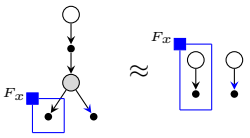


(6.5)

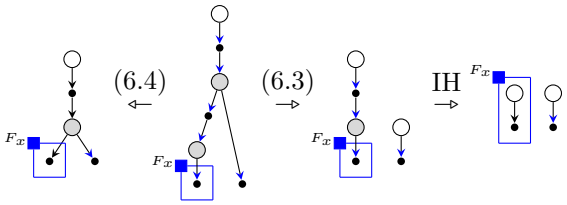
The base case is just (6.2), and so is trivially satisfied. Now we need to show that adding



to the other equations allows us to derive

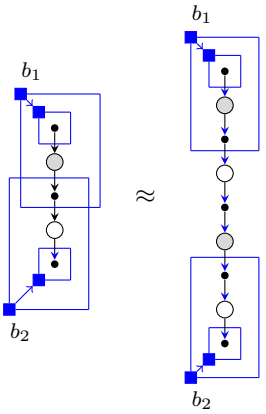


This follows using !-graph rewriting:

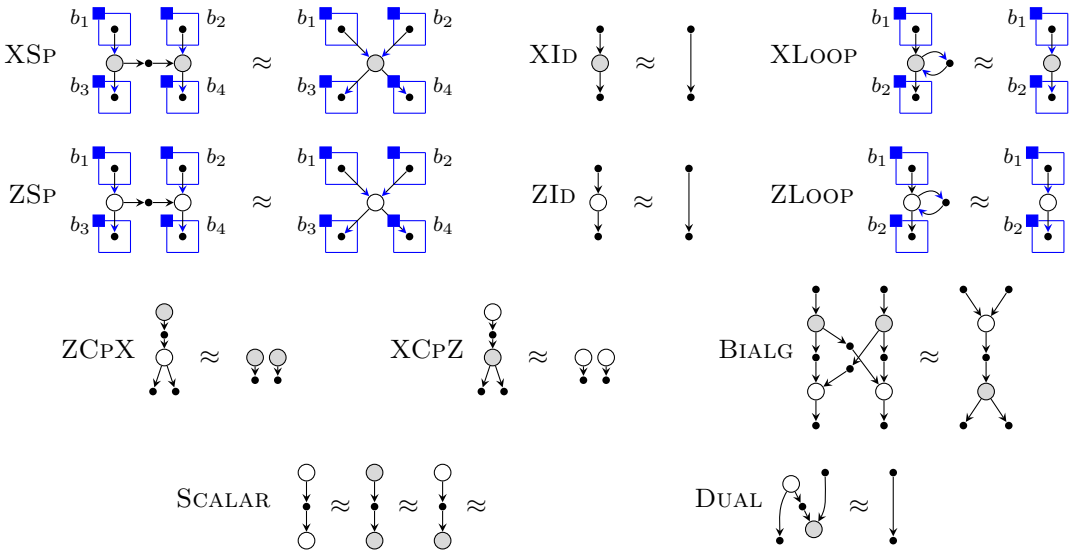


6.4.1 The Generalised Bialgebra Law

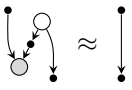
We now demonstrate a more ambitious application of !-box induction, where we derive the generalised bialgebra law for the Z/X calculus



Recall the axioms of the Z/X calculus:



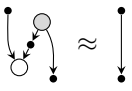
We have omitted the commutativity laws, as these are inherent in the fact that all the nodes are variable-arity, and hence the inputs (resp. outputs) of a Z or X node are indistinguishable from each other. In particular,



is actually the same string graph equation as Dual.

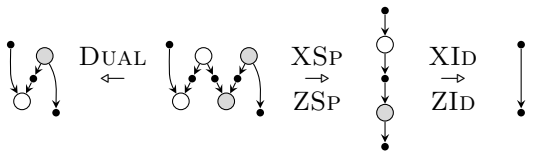
We will start by proving some symmetries of these equations. Note that, for brevity, we will tend to condense multiple applications of the same rewrite rule (or of rules that differ only by colour).

Lemma 6.4.3.



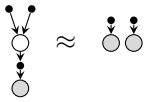
(6.6)

Proof.



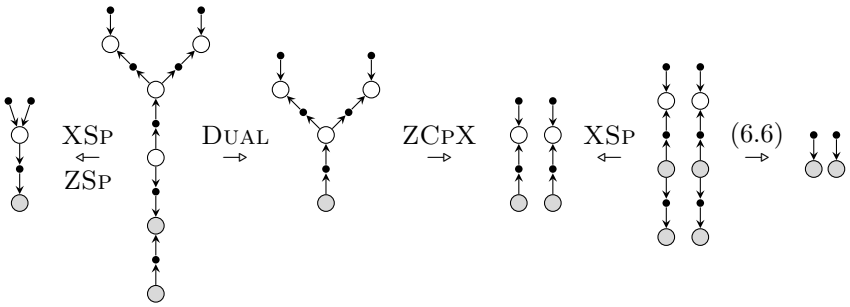
□

Lemma 6.4.4.



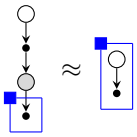
(6.7)

Proof.



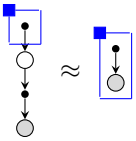
□

We have already derived



(6.8)

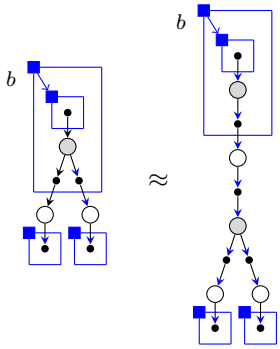
and



(6.9)

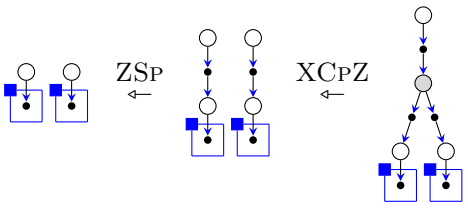
follows from (6.7) by essentially the same proof.

Lemma 6.4.5.

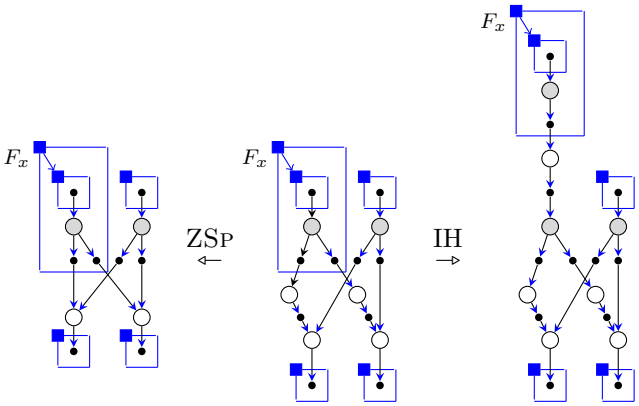


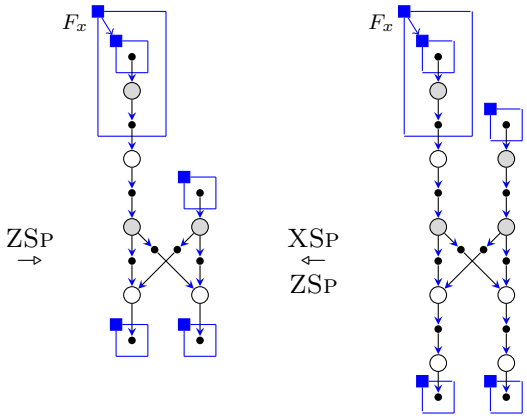
(6.10)

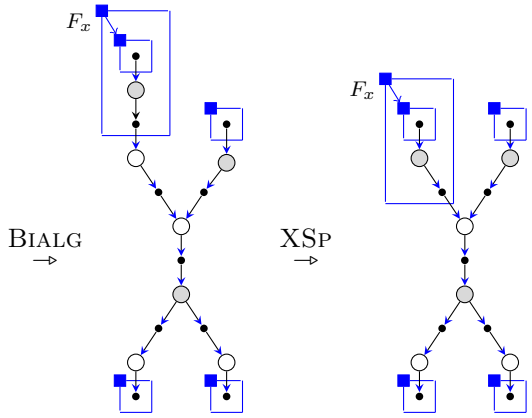
Proof. Proof by Induct on b. The base case is



and the inductive step is

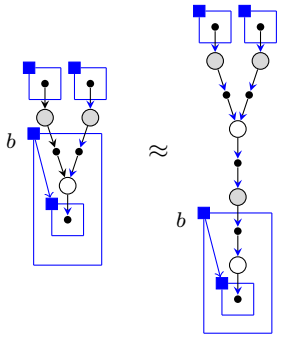






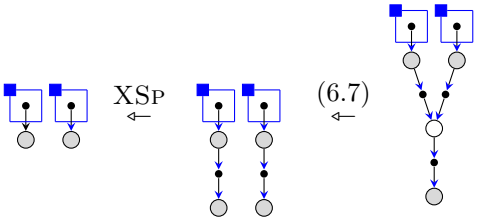
□

Lemma 6.4.6.

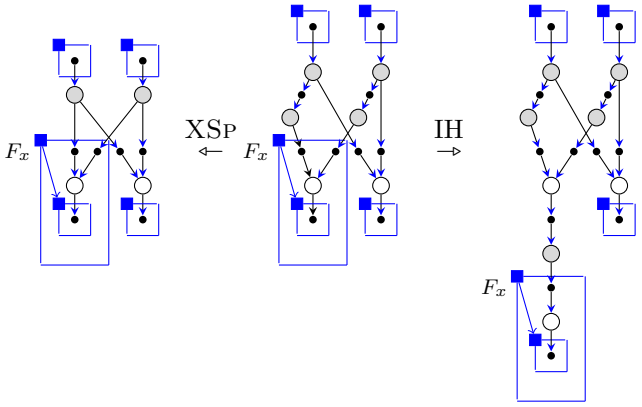


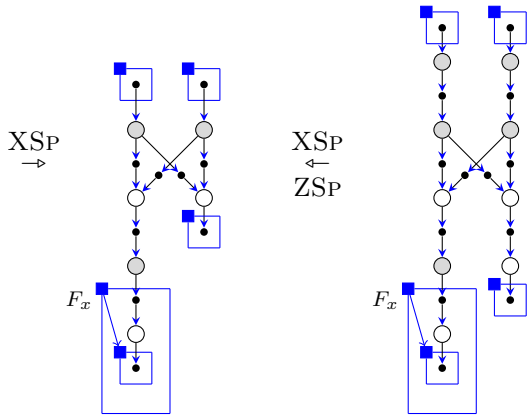
(6.11)

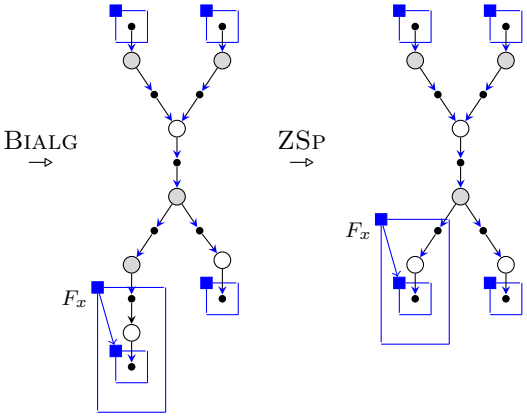
Proof. Proof by INDUCT on b. The base case is



and the inductive step is

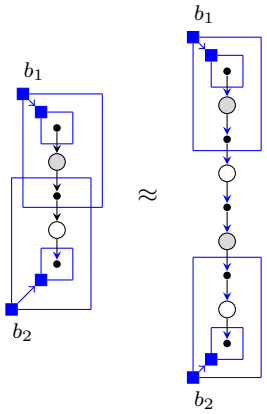




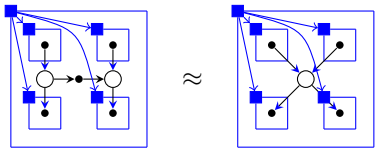


□

Theorem 6.4.7.

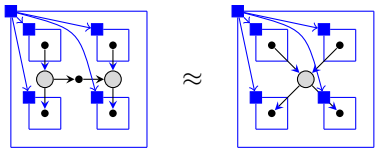


Proof. First we use Box on ZSp to get



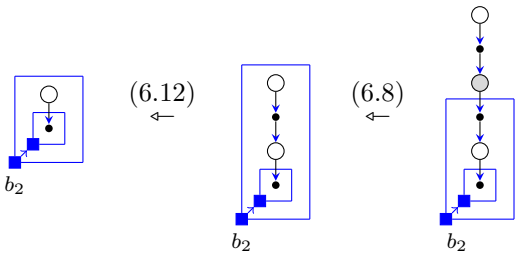
(6.12)

and on XSp to get

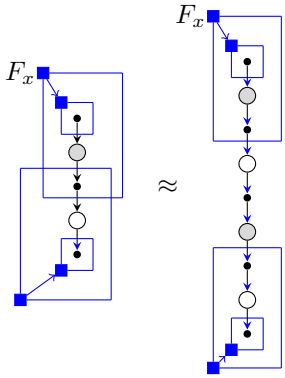


(6.13)

We will need to use Induct twice. First, we use it on b_1. The base case is

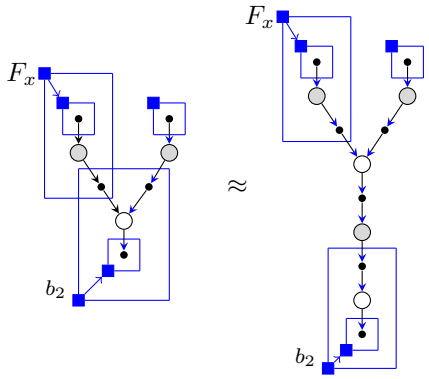


The inductive hypothesis is



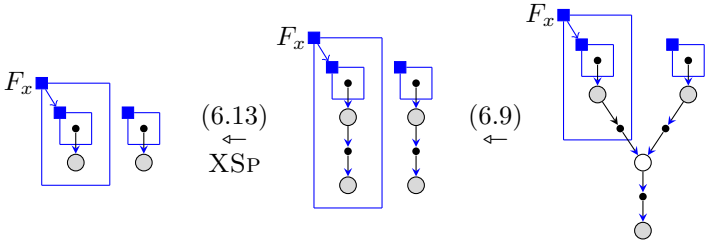
(6.14)

and we need to show

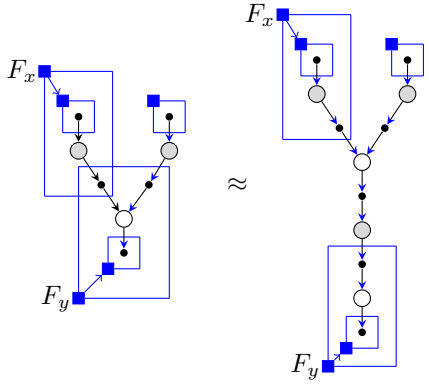


(6.15)

We apply INDUCT again, this time on b_2. The base case is

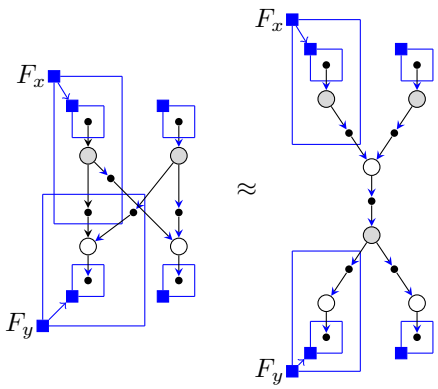


The new inductive hypothesis is



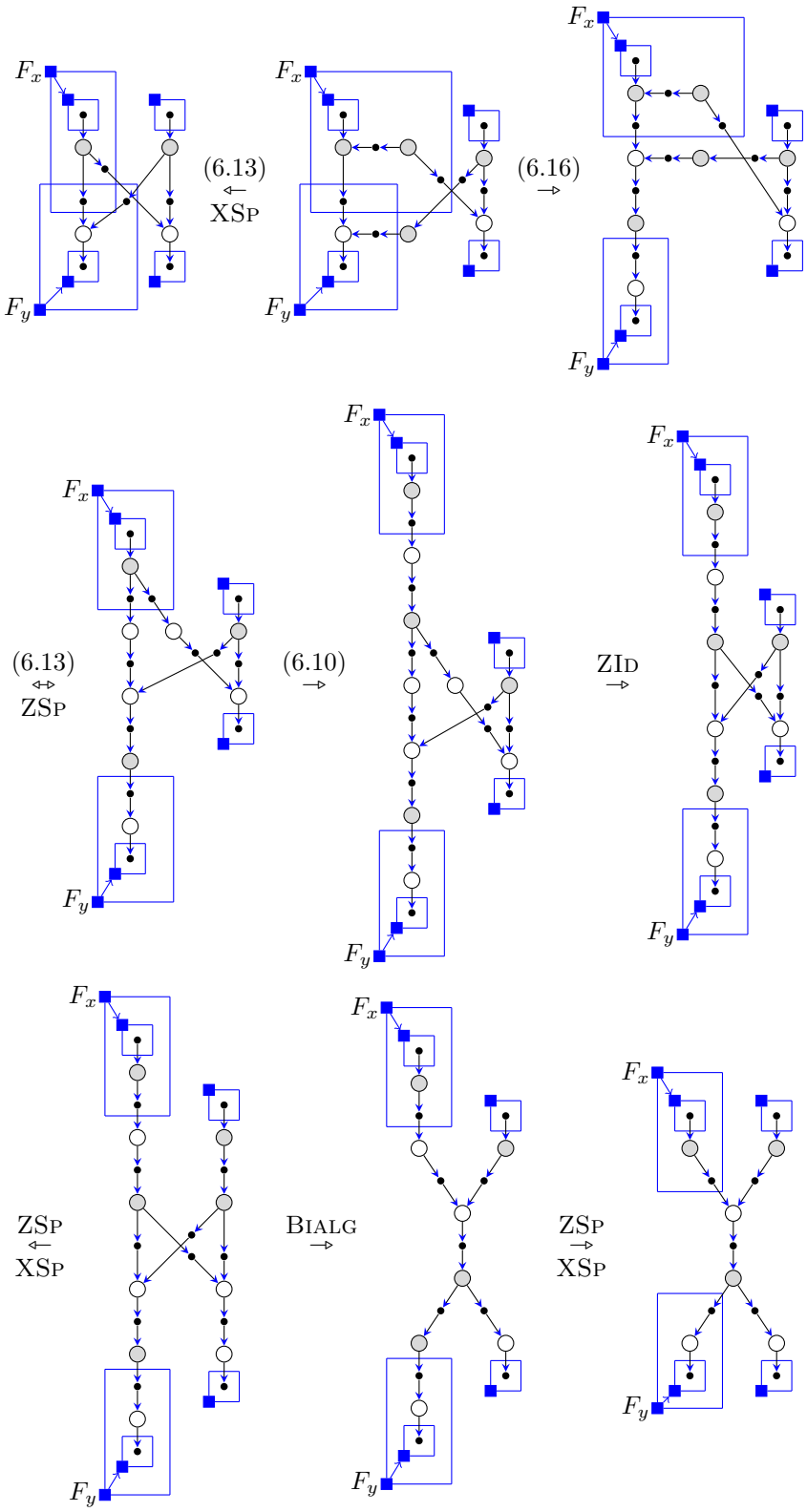
(6.16)

and we need to show



(6.17)

This is done as follows:



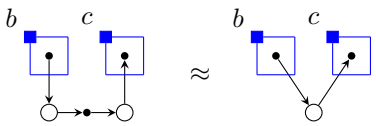
Thus we have demonstrated that (6.17) holds, so INDUCT gives us that (6.15) holds, and INDUCT again gives us that the theorem holds. □

This example, while a demonstration of the power of INDUCT, still requires the rewrite system to initially contain the spider laws. Appendix B demonstrates a possible way of using !-box induction to introduce !-boxes into rewrite systems that previously contained only concrete rewrite rules.

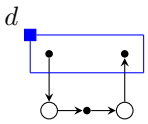
6.5 Merging !-boxes

The MERGE operation presented in this section was included as one of the core !-box operations in [26] (pp 6-7,10-11). Theorem 6.5.5 demonstrates that this is unnecessary.

Suppose we have the equation



and the graph



It is easy to see that for any concrete instance of the graph, there will be a concrete instance of the equation whose LHS matches the graph: we just need to do the same things to both b and c that we do to d. However, the LHS of the equation as it stands will not match the graph.

In order to allow this, we will introduce a new !-box operation that will merge two disjoint !-boxes in a !-graph. The two !-boxes will need to have the same parents in order to ensure that combining them will not affect the contents of any other !-box in the graph.

Definition 6.5.1 (MERGE; [26], pp 7)**.** Suppose G is a !-graph and $b, c \in !(G)$, with $B^{\uparrow}(b) \setminus b = B^{\uparrow}(c) \setminus c$ and $B(b) \cap B(c) = \{\}$, then $\mathrm{MERGE}_{b,c}(G)$ is a quotient of G where $B^{\uparrow}(b)$ and $B^{\uparrow}(c)$ are identified. More explicitly, this is the coequaliser

$$B^{\uparrow}(b) \underset{\widehat{c}}{\overset{\widehat{b}}{\rightrightarrows}} G \longrightarrow \mathrm{MERGE}_{b,c}(G) \tag{6.18}$$

in $\mathbf{Graph}/\mathcal{G}_{T!}$ where $\widehat{b}$ is the normal inclusion map and $\widehat{c}$ is the inclusion of $B^{\uparrow}(c)$ into G composed with the obvious isomorphism from $B^{\uparrow}(b)$ to $B^{\uparrow}(c)$.

This construction identifies b and c, and leaves all the other vertices of G untouched. The preconditions ensure that every incoming edge of b corresponds exactly to an incoming edge of c (and vice versa); each such edge to b is identified with its corresponding edge to c. If a pair of !-vertices satisfy the preconditions of MERGE, we say they are *mergable*.

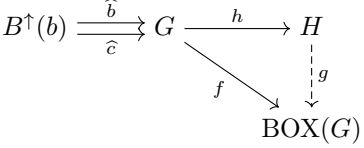
Proposition 6.5.2. *Let G be a !-graph with mergable !-vertices b and c. Then $\mathrm{MERGE}_{b,c}(G)$ is a !-graph and $U(G) \cong U(\mathrm{MERGE}_{b,c}(G))$.*

Proof. Let $H = \text{MERGE}_{b,c}(G)$, and let h be the coequaliser map.

$$B^{\uparrow}(b) \underset{\widehat{c}}{\overset{\widehat{b}}{\rightrightarrows}} G \xrightarrow{h} H$$

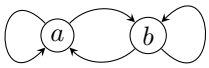
H exists in $\mathbf{Graph}/\mathcal{G}_{T!}$, as $\mathbf{Graph}/\mathcal{G}_{T!}$ has coequalisers of monomorphisms. Note that h is surjective by lemma 4.2.7 since it is a regular, and hence strong, epimorphism of $\mathbf{Graph}/\mathcal{G}_{T!}$.

We can construct a morphism $f : G \to \text{BOX}(U(G))$ that is the identity on $U(G)$ (and collapses all the other !-boxes), and this trivially coequalises $\widehat{b}$ and $\widehat{c}$. So there is a unique g making



commute. Since $U(f)$ is an isomorphism, $U(h)$ is a (split) monomorphism as well as being a strong epimorphism, and hence is invertible. So $U(G) \cong U(H)$ and hence $U(H)$ must be a string graph, since $U(G)$ is.

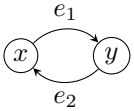
Let x, y be vertices in $!(G)$ such that $h(x) = h(y)$. We can show that either $x = y$ or $\{x, y\} = \{b, c\}$. Suppose $x \neq y$, let G' be $U(G)$ together with the graph of !-vertices



and edges from a and b to each vertex of $U(G)$. Then let $f : G \to G'$ be the identity on $U(G)$ and take every vertex in $!(G)$ to a except y, which it takes to b. This also fixes the edge maps. If $y \neq b$ and $y \neq c$, g will coequalise $\widehat{b}$ and $\widehat{c}$, and so there must be a unique map $g : H \to G'$ such that $f = g \circ h$. But then $g(h(x)) \neq g(h(y))$ even though $h(x) = h(y)$, which is impossible. So we must have $y = b$ or $y = c$. Repeating with the roles of x and y switched, we get that $x = b$ or $x = c$. Thus if $x \neq y$, then $\{x, y\} = \{b, c\}$.

$\beta(H)$ is reflexive as it is the image of $\beta(h)$, the domain of which is reflexive.

Let $x \neq y$ be vertices in $\beta(H)$ with e_1, e_2 edges between them such that $s(e_1) = t(e_2) = x$ and $t(e_1) = s(e_2) = y$.



$\beta(G)$ is anti-symmetric, and so the preimages of e_1 and e_2 in under h cannot be in the same loop construction. Thus either x or y must be mapped to by more than one vertex of G, and hence must have the preimage $\{b, c\}$. This cannot be true of both, since $x \neq y$. WLOG, suppose it is true of x. Then y', the preimage of y, must be in $B(b)$ and $B^{\uparrow}(c)$ (or vice versa). But $B^{\uparrow}(b) \setminus b = B^{\uparrow}(c) \setminus c$

and (since $y' \neq c$) y' must be in $B^{\uparrow}(b)$, which can only be true if $y' = b$, which we assumed was not the case. So there can be no such e_1 and e_2.

Simplicity of $\beta(H)$ follows a similar argument: if e_1 and e_2 are edges from x to y where $x \neq y$, then if the preimage of y is $\{b, c\}$, x must be in the image of $\widehat{b}$ and $\widehat{c}$ and so the preimages of e_1 and e_2 must be coequalised by h and hence $e_1 = e_2$, and if the preimage of x is $\{b, c\}$ then the fact that $B(b) \cap B(c) = \varnothing$ means the preimages of e_1 and e_2 must be the same, and hence $e_1 = e_2$.

For transitivity, let x, y, z be vertices in $\beta(H)$ with e_1 an edge from x to y and e_2 from y to z. Let e_1', e_2' be arbitrary preimages of the respective edges under h, as before. Then either $t(e_1') = s(e_2')$, in which case the existence of an edge from x to z is immediate from transitivity of G, or $\{t(e_1'), s(e_2')\} = \{b, c\}$. WLOG, assume $t(e_1') = b$ and $s(e_2') = c$. Then $B^{\uparrow}(b) \setminus b = B^{\uparrow}(c) \setminus c$ gives us that $s(e_1')$ must be a predecessor of c, and so x must be a predecessor of z, as required.

Let $d \in !(H)$. Then, because h is surjective and $U(h)$ is bijective, $U(B(d))$ is the union of $U(h)[U(B(d'))]$ for each preimage d' of d under h. But $U(h)$ is an isomorphism, so $U(B(d))$ is the union of open subgraphs of $U(H)$, and hence is open in $U(H)$.

For the final requirement of a !-graph, suppose $d, d' \in !(H)$ with $d' \in B(d)$. We need to show, for any vertex v of H with $v \in B(d')$, that $v \in B(d)$. If v is a !-vertex, this follows from transitivity of $\beta(H)$. Otherwise, the edge e from d' to v has a preimage e_0 under h, as does the edge e' from d to d'. If $t(e_0') = s(e_0)$, there must be an edge from $s(e_0')$ to $t(e_0)$ in G, which will map to an edge from d to v in H. Otherwise, we must have $t(e_0') = b$ and $s(e_0) = c$ (or vice versa, which is equivalent). But then the preconditions for MERGE require an edge from $s(e_0')$ to c as well, which also induces an edge from $s(e_0')$ to $t(e_0)$, and hence from d to v. □

Proposition 6.5.3. *Let $f : G \to H$ be a morphism of $\mathbf{BGraph}_T$ that reflects !-box containment, and let b and c be mergable !-vertices of G such that $f(b)$ and $f(c)$ are mergable in H. Then there is a unique morphism*

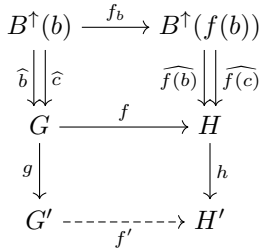
$$\mathrm{MERGE}_{f(b),f(c)}(f) : \mathrm{MERGE}_{b,c}(G) \to \mathrm{MERGE}_{f(b),f(c)}(H)$$

making the following diagram commute

$$\begin{array}{ccc} G & \xrightarrow{\quad f \quad} & H \\ \downarrow & & \downarrow \\ \mathrm{MERGE}_{b,c}(G) & \xrightarrow[\mathrm{MERGE}_{f(b),f(c)}(f)]{} & \mathrm{MERGE}_{f(b),f(c)}(H) \end{array}$$

where the down arrows are the coequaliser maps from the definition of MERGE*, and this morphism reflects !-box containment. Further, this construction preserves monomorphisms and strong epimorphisms and respects composition.*

Proof. Let $G' = \text{MERGE}_{b,c}(G)$ and $H' = \text{MERGE}_{f(b),f(c)}(H)$. We will construct the following commuting diagram



where f' will be uniquely determined by universality of coequalisers. This will then be $\text{MERGE}_{f(b),f(c)}(f)$.

The top square is two separately-commuting squares: $f \circ \widehat{b} = \widehat{f(b)} \circ f_b$ and $f \circ \widehat{c} = \widehat{f(c)} \circ f_b$. If d is a vertex in $B^\uparrow(b)$, then the edge witnessing this is mapped by f to one witnessing $f(d) \in B^\uparrow(f(b))$. So if we let f_b be $f \circ \widehat{b}$, the restriction of f to $B^\uparrow(b)$, reinterpreted with codomain $B^\uparrow(f(b))$ then we have the first commuting square. The second follows from the fact that $B^\uparrow(b) \setminus b = B^\uparrow(c) \setminus c$ and similarly for $f(b)$ and $f(c)$.

Now h coequalises $\widehat{f(b)} \circ f_b$ and $\widehat{f(c)} \circ f_b$, and hence coequalises $f \circ \widehat{b}$ and $f \circ \widehat{c}$. So $h \circ f$ coequalises $\widehat{b}$ and $\widehat{c}$. Then universality of g means that there is a unique f' making the bottom square of the diagram commute. This universality also means that MERGE respects composition.

The fact that f reflects !-box containment means that f_b is surjective, and hence epic. Then if we have morphisms $a : G' \to K$ and $b : H \to K$ such that $a \circ g = b \circ f$, we know

$$\begin{aligned} & a \circ g \circ \widehat{b} = a \circ g \circ \widehat{c} \\ \Rightarrow\ & b \circ \widehat{f(b)} \circ f_b = b \circ \widehat{f(c)} \circ f_b \\ \Rightarrow\ & b \circ \widehat{f(b)} = b \circ \widehat{f(c)} \end{aligned}$$

Then, since b equalises $\widehat{f(b)}$ and $\widehat{f(c)}$, there is a unique arrow $k : H' \to K$ such that $b = k \circ h$. Since g is epic, we also have $a = k \circ f'$, and hence the bottom square of the diagram is a pushout. $\mathbf{BGraph}_T$ preserves monomorphisms under pushouts, and hence f' is monic whenever f is. Also, if f is strongly epic, and hence surjective, the same must be true of $h \circ f$ and hence of $f' \circ g$. So f' is surjective, and hence strongly epic. Since it is surjective, it trivially reflects !-box containment. □

Since $U(G) \cong U(\text{MERGE}(G))$ for !-graphs, $U(f)$ is the same as $U(\text{MERGE}(f))$ (up to isomorphism) and so we can simply apply MERGE to the morphisms of definition 5.1.2 to get the following:

Corollary 6.5.4 ([26], pp 10-11)**.** *Let $L \xleftarrow{i} I \xrightarrow{j} R$ be a !-graph equation, and let b, c be mergable !-vertices of I. Then*

$$\text{MERGE}_{i(b),i(c)}(L) \xleftarrow{\text{MERGE}_{i(b),i(c)}(i)} \text{MERGE}_{b,c}(I) \xrightarrow{\text{MERGE}_{j(b),j(c)}(j)} \text{MERGE}_{j(b),j(c)}(R)$$

is also a !-graph equation.

In the case that b and c are mergable !-vertices of $G \approx H$, we posit the following rule:

$$\textsc{Merge}\ \frac{E \vdash G \approx H}{E \vdash \mathrm{MERGE}_{b,c}(G \approx H)}$$

Theorem 6.5.5. Merge *is sound.*

Proof. We will show that every concrete instance $G' \approx H'$ of $\mathrm{MERGE}_{b,c}(G \approx H)$ is also a concrete instance of $G \approx H$. Suppose S is a concrete instantiation of $G' \approx H'$ from $\mathrm{MERGE}_{b,c}(G \approx H)$ in expansion-normal form.

While instantiations may only contain COPY, DROP or KILL operations (EXP being a composite of these), for the purposes of this argument we will relax this to allow MERGE operations as well. This is purely a convenience to allow us to call our intermediate steps "instantiations" before we finally arrive at a genuine instantiation containing only those three operations. So we start with the "instantiation" $\mathrm{MERGE}_{b,c}; S$ of $G' \approx H'$ from $G \approx H$, and transform it into a genuine instantiation with no MERGE operations.

The intermediate instantiations may have multiple MERGE operations, but we will always keep these together, and they will not interfere with each other: if $H = \mathrm{MERGE}_{b,c}(G)$ and $K = \mathrm{MERGE}_{b',c'}(H)$ are two adjacent operations in the instantiation, where $G \xrightarrow{g} H \xrightarrow{h} K$ are the coequaliser maps, we will have that $b' \notin B^{\uparrow}(g(b))$ and $g(b) \notin B^{\uparrow}(b')$, and similarly $h(g(b))$ will not interfere with the !-vertices acted on by any MERGE operation on K and so on. This allows us to freely reorder the MERGE operations.

We will use this to collate them into a "composite" MERGE that simultaneously operates on multiple pairs of mergable !-vertices, where each pair has distinct parents to any other pair. We will only ever have one such operation in the sequence; initially, it will be at the start and will have a single pair, $\langle b, c\rangle$. When it has no pairs, there are no MERGE operations left in the sequence and we are done.

We will move the MERGE right one operation at a time until the set of pairs it operates on is empty. So consider the operation to its right. This is either KILL_d or EXP_d for some !-vertex d. d cannot be a child !-vertex of any of the merged vertices, since d must be a top-level !-vertex. If none of the merged vertices are in $B(d)$, the operations are independent, and we can move the MERGE one place to the right without changing the resulting graph.

Suppose the operation is KILL_d. If d is a merged vertex, call its preimages under the coequaliser d_0 and d_1. If we remove the KILL_d, remove the pair $\langle d_0, d_1\rangle$ from the MERGE and put $\mathrm{KILL}_{d_0}; \mathrm{KILL}_{d_1}$ before the MERGE, we will get the same graph. Otherwise, suppose d is not a merged vertex, but $B(d)$ contains a merged vertex. Then we can swap KILL_d with the MERGE and remove any pairs in $B(d)$ from the MERGE.

Suppose instead the operation is EXP_d. If d is a merged vertex, we can remove EXP_d and put $\mathrm{EXP}_{d_0}; \mathrm{EXP}_{d_1}$ before the MERGE, leaving the set of merged vertices untouched. Otherwise,

suppose d is not a merged vertex but $B(d)$ contains one, and say it results from the pair $\langle x, y\rangle$. Then we can swap EXP_d and the MERGE, providing we add $\langle x', y'\rangle$ to the set of merged vertices, where x' is the copy of x under the expansion and y' is the copy of y. x' and y' have no edges to or from x and y, or the !-vertices of any other merged pair. If there are any other such pairs, do the same to them as well.

Since the number of operations to the right of the MERGE always decreases, this will eventually terminate, and we will have a concrete of instantiation of $G' \approx H'$ from $G \approx H$. □

6.6 A More Traditional Logic

Those familiar with formal logics will have noticed something strange about the one presented at the start of this chapter, as well as the one in section 3.5: the equality predicate $\approx$ is not really a predicate at all. What we have presented is a logic of "string graph equations" and "!-graph equations", where those "equations" are actually spans in a category.

In this section, we will sketch out an equational logic of !-graphs, where the equality predicate is just a formal symbol indicating that the objects on either side of it should be considered equal. As we noted in section 3.4.1, we need to somehow encode the correlation between boundaries of string graphs; we also need to encode the correlation between !-vertices in both graphs. !-graph equations are one way of doing this, but to get a more traditional equality predicate, we will extend the idea of a framed cospan (definition 3.4.1) to !-vertices.

Definition 6.6.1 (!-Framed Cospan)**.** A !-*graph frame* is a triple $(X, <, \mathrm{sgn})$ where X is a !-graph consisting only of isolated wire-vertices, !-vertices and edges whose sources are !-vertices; $<$ is a total order on V_X, the vertices of X; and $\mathrm{sgn} : V_{U(X)} \to \{+, -\}$ is the *signing map*.

A cospan of !-graph monomorphisms $X \xrightarrow{d} G \xleftarrow{c} Y$ is called a !-*framed cospan* if

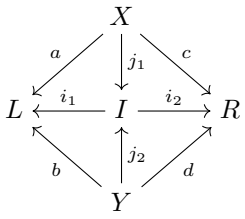
1. X and Y are !-graph frames
2. G contains no isolated wire-vertices
3. the following is a pushout square:

$$\begin{array}{ccc} \beta(G) & \hookrightarrow & \mathrm{im}(d) \\ \downarrow & & \downarrow \\ \mathrm{im}(c) & \hookrightarrow & \mathrm{Bound}_!(G) \end{array}$$

4. for every $v \in V_{U(X)}$, $d(v) \in \mathrm{In}(G) \Leftrightarrow \mathrm{sgn}(v) = +$
5. for every $v \in V_{U(Y)}$, $c(v) \in \mathrm{Out}(G) \Leftrightarrow \mathrm{sgn}(v) = +$

As with the framed cospans of string graphs, we can frame a !-graph equation:

Definition 6.6.2. A *framing* of a !-graph equation $L \approx_{i_1,i_2} R$ is a pair of !-framed cospans $X \xrightarrow{a} L \xleftarrow{b} Y$ and $X \xrightarrow{c} R \xleftarrow{d} Y$ such that there are morphisms j_1 and j_2 forming the coproduct (ie: disjoint union) $X \xrightarrow{j_1} I \xleftarrow{j_2} Y$ and making the following diagram commute:



Just as with framings of !-graph equation, such a framing is equivalent to a !-graph equation, in the sense that any !-graph equation can be framed, and the !-graph equation can be recovered from the two !-framed cospans. If two !-framed cospans have isomorphic left and right (or domain and codomain) !-graph frames, they induce a !-graph equation, and we call them *compatible*.

With this is mind, we can construct a logic of !-framed cospans that is equivalent to the logic presented in this chapter. We have a choice in how we represent the set of axioms: we can use a set of !-graph equations and the rule

$$(\textsc{Axiom})\ \frac{G \approx_{i,j} H \in E}{E \vdash \hat{G} \approx \hat{H}}$$

where $(\hat{G}, \hat{H})$ is a framing of $G \approx_{i,j} H$, or we can use a set of pairs of compatible !-framed cospans and have the rule

$$(\textsc{Axiom})\ \frac{(\hat{G}, \hat{H}) \in E}{E \vdash \hat{G} \approx \hat{H}}$$

The equivalence relation rules are simple:

$$(\textsc{Refl})\ \frac{}{E \vdash \hat{G} \approx \hat{G}} \qquad (\textsc{Sym})\ \frac{E \vdash \hat{G} \approx \hat{H}}{E \vdash \hat{H} \approx \hat{G}} \qquad (\textsc{Trans})\ \frac{E \vdash \hat{G} \approx \hat{H} \qquad E \vdash \hat{H} \approx \hat{K}}{E \vdash \hat{G} \approx \hat{K}}$$

For the others, we will need to introduce some new notation, or at least extend existing notation. For example, if $f : G \to H$ is a wire homeomorphism of !-graphs, and $\hat{G} = X \to G \leftarrow Y$ is a framed cospan, then there is a framed cospan $\hat{H} = X \to H \leftarrow Y$ where the inclusions of the framed cospans commute with $f_!$ and f_B. We can then view f as a wire homeomorphism from $\hat{G}$ to $\hat{H}$.

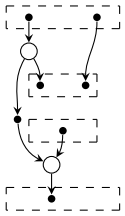
Homeo can then be expressed as

$$(\textsc{Homeo})\ \frac{E \vdash \hat{G} \approx \hat{H}}{E \vdash \hat{G} \approx f(\hat{H})}$$

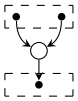
where f is a wire homeomorphism (note that the presence of Sym means that this is equivalent to the version of Homeo on !-graph equations).

Leibniz requires a bit more work. We will borrow the notion of *contexts* from term-rewriting, where they are terms with "holes" in them: placeholders that can be substituted with other terms to produce a valid term overall.

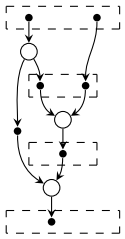
The idea with !-graphs will be to provide both an "external" interface to the graph, as a !-framed cospan does, and an "internal" interface. An simple example (without any !-boxes) would be



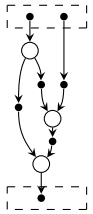
We could then take a !-framed cospan whose frames match the internal interface of the context, such as



and embed it in the context by merging the images of the frames



producing the !-framed cospan



When there are !-vertices to account for, the !-graph frames of the external interface must contain all the !-vertices (and edges between them) in order for the final construction to be a !-framed cospan. The internal interface need not contain all the !-vertices of the graph, though.

Definition 6.6.3. A !-*graph context* C is a pair of !-graph cospans

$$X_I \xrightarrow{d_I} G \xleftarrow{c_I} Y_I$$

$$X_O \xrightarrow{d_O} G \xleftarrow{c_O} Y_O$$

where

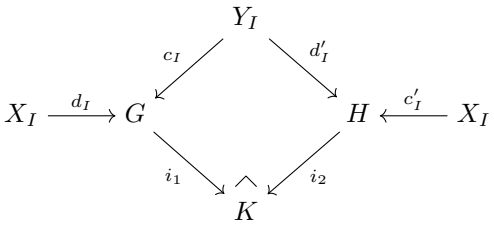
- G contains no isolated wire-vertices
- X_I, X_O, Y_I and Y_O are !-graph frames
- $\beta(X_I) \cong \beta(Y_I)$
- $\beta(X_O) \cong \beta(Y_O) \cong \beta(G)$

- the morphisms are all monic and reflect !-box containment
- $\mathrm{Bound}(U(G))$ is the disjoint union of the images of $U(d_I)$, $U(c_I)$, $U(d_O)$ and $U(c_O)$
- for every $v \in V_{U(X_I)}$, $d_I(v) \in \mathrm{In}(G) \Leftrightarrow \mathrm{sgn}(v) = +$, and similarly for d_O
- for every $v \in V_{U(Y_I)}$, $c_I(v) \in \mathrm{Out}(G) \Leftrightarrow \mathrm{sgn}(v) = +$, and similarly for c_O

Given such a !-graph context and another !-framed cospan

$$\hat{H} \;=\; Y_I \xrightarrow{d'_I} H \xleftarrow{c'_I} X_I$$

we can merge the context with $\hat{H}$ (we say that C *accepts* $\hat{H}$). We first merge G and H, using Y_I as the overlap, in a manner similar to the composition of framed cospans from section 3.4.1:



K is a !-graph by a similar argument to the one that shows composition of framed cospans to be well-defined (note the reversal of the frames for $\hat{H}$ means that the interpretation of the signing map on the frames is reversed, so inputs of H will be merged with outputs of G and vice versa).

Next, we merge the two images of X_I using the following coequaliser:

$$X_I \underset{i_2 \circ c'_I}{\overset{i_1 \circ d_I}{\rightrightarrows}} K \dashrightarrow^{f} G + H$$

Then we define $C[\hat{H}]$ to be the framed cospan

$$X_O \xrightarrow{f \circ i_1 \circ d_O} G + H \xleftarrow{f \circ i_1 \circ c_O} Y_O$$

Now we can write Leibniz as

$$(\textsc{Leibniz})\quad \frac{E \vdash \hat{G} \approx \hat{H}}{E \vdash C[\hat{G}] \approx C[\hat{H}]}$$

where C is a !-graph context that accepts $\hat{G}$ and $\hat{H}$ (note that the compatibility of $\hat{G}$ and $\hat{H}$ means that if C accepts one then it must accept the other).

In this version of Leibniz, the context graph takes the place of D in the !-graph rewrite

$$\begin{array}{ccccc} G & \longleftarrow & I & \longrightarrow & H \\ \downarrow & & \downarrow & & \downarrow \\ G' & \longleftarrow & D & \longrightarrow & H' \end{array}$$

and the internal frames (X_I and Y_I) take the place of I.

The final piece of this logic of !-graph frames is the !-box operations. We extend them to !-graph cospans in the same way that we extended them to !-graph equations: if

$$\hat{G} \;=\; X \xrightarrow{d} G \xleftarrow{c} Y$$

then

$$\mathrm{OP}(\hat{G}) \;=\; \mathrm{OP}(X) \xrightarrow{\mathrm{OP}(d)} \mathrm{OP}(G) \xleftarrow{\mathrm{OP}(c)} \mathrm{OP}(Y)$$

Now we can write, for example,

$$(\textsc{Copy})\ \frac{E \vdash \hat{G} \approx \hat{H}}{E \vdash \mathrm{COPY}_b(\hat{G}) \approx \mathrm{COPY}_b(\hat{H})}$$

(since the compatibility of $\hat{G}$ and $\hat{H}$ means that we consider b to be in both if it is in either), and similarly for the other !-box operation rules.

Thus we have a logic with a normal equality predicate based on !-framed cospans. What is more, by considering only concrete graphs, the same construction can be used to produce a logic based on framed cospans from the logic presented in section 3.5.

In practice, a theorem prover would probably use names to store the equivalent of the !-graph frames; the signing maps and the division between the domain and codomain are not particularly important for the construction of proofs (as can be seen from the fact that the !-graph equation structures do not store this information).

Chapter 7

Computability of !-Graph Matching

Suppose we have a !-graph (or string graph) G and a !-graph rewrite rule $L \multimap R$ that we wish to use to rewrite G; as previously stated, we want rewriting to be up to wire homeomorphism. If we can find a rewrite rule $L' \multimap R'$ that is wire-homeomorphic to an instance of $L \multimap R$ such that there is a local isomorphism $m : L' \to G$ that reflects !-box containment, theorem 5.3.3 tells us that the rewrite

$$\begin{array}{ccccc} L' & \xleftarrow{i'} & I' & \xrightarrow{j'} & R' \\ {\scriptstyle m}\downarrow & & \downarrow{\scriptstyle d} & & \downarrow{\scriptstyle r} \\ G & \xleftarrow[g]{} & D & \xrightarrow[h]{} & H \end{array} \tag{7.1}$$

exists. In fact, it is easy to find this rewrite: if we remove from G the image of everything in the *interior* of L (ie: the image of everything in L that is not in the image of i), we get D; g is the subgraph relation and d is a restriction of m. H, being a pushout of monomorphisms, is just a graph union. In a practical implementation, where vertices and edges are likely to be named components, the main difficulty is managing the names to make sure there are no clashes when building H.

That still leaves finding suitable instances, finding suitable rules wire homeomorphic to those instances and finding a local isomorphism that reflects !-box containment. Section 3.6.2 discussed the latter two (except reflecting !-box containment, which can be implemented by filtering the results); in this chapter, we will demonstrate that, providing $L \multimap R$ is "well behaved" in terms of the instances it can produce, we can finitely enumerate all instances of the rule that can result in a matching onto a particular G. We will also describe how Quantomatic implements these steps.

7.1 Enumerating Instances

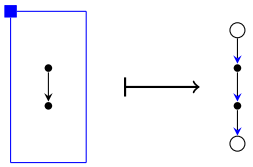
In order to find all possible matchings, we need a way of searching the space of graphs H such that $G \succeq H$; while this space will almost always be infinite, it will usually be the case that only a finite number of them will be candidates for matchings.

Suppose we wish to match a !-graph G onto another !-graph H; we need to find an instance G' of G and a monomorphism from G' to H (or, rather, from a graph wire homeomorphic to G' to a

graph wire homeomorphic to H). The monomorphism must satisfy additional constraints (definition 5.3.2), but these are not important to this discussion.

As long as G has any (unfixed) !-vertices, it will have an infinite number of instances (to see this, consider repeatedly applying COPY to one of the !-vertices). However, in most cases, only a finite number of these will have a monomorphism to H.

In particular, we can safely ignore any instance of G that has more node-vertices, circles or fixed !-vertices than H, as the number of each of these is constant under wire homeomorphism. We cannot, however, restrict the search based on the number of wire-vertices. Consider, for example, the following pattern and target graphs:



No matter how many times the !-box in the pattern is expanded, we can always introduce enough wire-vertices in the target graph's wire to accommodate them all. So there are an infinite number of matchings in this case. Empty !-boxes do not even need a wire in the target graph to exhibit this behaviour.

We call these (unfixed) !-boxes, containing no node-vertices or circles, *wild* !-boxes. In order for it to be possible to enumerate all candidate instances of G, we will require that no instance of G contains any wild !-boxes. To show this for any given G, it is sufficient to show that for each $b \in !(G)$, $B(b)$ is not wild after killing all !-vertices not in $B^{\uparrow}(b)$. The rest of this discussion will assume no instance of G contains wild !-boxes.

The first step is to consider how to explore the space of instances in an ordered way. There may be an infinite number of witnesses of even a single instance, since applying $\mathrm{COPY}_b; \mathrm{KILL}_{b^1}$ to a graph G with a !-vertex b yields a graph isomorphic to G, and so we cannot simply try to build every possible instantiation. At the same time, we cannot depend on expansion-normal form as we need to deal with arbitrary instances, not just concrete ones.

That said, we can produce something similar enough to expansion-normal form for our purposes. We will use the FIX operation used in section 6.4, and we will adjust our definition of depth to ignore !-vertices that have been fixed (so a !-vertex with only fixed parents has depth 0). Since we are really using this as a notational convenience, we will relax the constraints on FIX to allow the fixing of !-vertices whose parents are all fixed. The important thing to note is that, in the context of instantiations, FIX behaves just like DROP: after FIX_b^x, there can be no further operations on b, and it can potentially reduce the depth of any children of b.

If we take an instantiation S of G' from G' and a fresh fixing tag x, and append a FIX_b^x operation for every $b \in !(G')$, then the result, call it S', is enough like a concrete instantiation for the arguments

from section 6.2 to hold with minimal adjustment (just read "FIX^x or DROP" wherever DROP is mentioned). We also need an equivalent of EXP$_b$; we will call this CFIX^x_b, which will be a shorthand for COPY$_b$; FIX$^x_{b^1}$.

So we can transform S' into a witness for $G \succeq G'$ that is depth-ordered (according to our revised definition of depth) and is composed entirely of EXP, KILL and CFIX^x operations. As a result, we only need to search for instantiations of this form in order to find all possible instances.

Remark 7.1.1. Using this trick comes with a caveat: when we attempt to find a morphism from G' to H, an x-fixed !-vertex would ordinarily only be allowed to match another x-fixed vertex. However, we want x-fixed vertices to behave like unfixed vertices when it comes to building matching morphisms, and allow them to match *any* vertex of the target graph. This can be acheived by either removing all x-fixing tags from G' before finding the matching morphism (which is the approach taken by the algorithm in section 7.2) or by relaxing the constraint with a special case for the x tag.

We will call the area of the pattern graph not in any !-boxes or only in fixed !-boxes the *match surface*. This is the graph that would result from killing all the unfixed !-boxes. Importantly, no EXP, KILL or CFIX^x operation can reduce the match surface. Furthermore, if every !-box of every instance of G contains at least one node-vertex or circle, EXP and CFIX^x must always increase the number of node-vertices or circles in the match surface.

We will take an inductive approach to finding instances of G that are candidates for a matching onto H, starting with the zero-length instantiation. To build the instantiations of length $n+1$, for each instantiation S of length n (resulting in a !-graph G_S) and each unfixed $b \in !(G_S)$ with depth 0, we will create three new instantiations by appending KILL$_b$, EXP$_b$ and CFIX^x_b to S. If the resulting graph has more node-vertices, circles or fixed !-vertices in its match surface than are in H, we will discard that instantiation (and hence not consider it as a prefix for the instantiations of length $n+2$).

Remark 7.1.2. Note that there are more efficient versions of the algorithm; for example, it is sufficient to choose an arbitrary depth 0 !-vertex at each stage rather than generating new instantiations for each such !-vertex. However, it is easier to see that the algorithm here is correct, and this version suffices to show computability.

It should be clear, given the preceeding discussion, that the algorithm will enumerate all instances of G that have a monomorphism to H. It remains for us to show that it will terminate, providing every !-box of every instance of G contains at least one node-vertex or circle.

Let x_H be the number of node-vertices in H and y_H the number of circles. Suppose S is an instantiation produced by the algorithm at some iteration step, and let x_S and y_S be the corresponding properties for the match surface of G_S. Every EXP and CFIX^x operation must increment

either x_S or y_S (and KILL cannot decrease either), and the algorithm will discard any instantiation where

$$x_S + y_S > x_H + y_H$$

So there can be at most $x_H + y_H$ EXP or CFIX^x operations in S.

Each of these operations can, at most, double the number of unfixed !-vertices in the graph, and KILL can only reduce that number. Thus if $z_G = |!(G)|$, S cannot contain more than $z_G \cdot 2^{x_H+y_H}$ KILL operations. $|S|$, the length of S, is therefore bounded by

$$x_H + y_H + z_G \cdot 2^{x_H+y_H}$$

which is a finite number fixed by G and H.

The same bound also places a limit on $|!(G_S)|$, limiting the number of ways S can be extended. The total number of steps in the algorithm is then bounded, and hence it terminates.

Note that this approach encompasses rewriting both string graphs and !-graphs with !-graph rewrite rules; the former is simply a special case of the latter, without any CFIX^x operations.

7.2 Matching in Quantomatic

Quantomatic interleaves the three steps of !-graph matching (instantiation, wire homeomorphism and building a suitable morphism) in a branching algorithm. In this section, we will describe this approach. We will start with the algorithm as currently implemented, which assumes the target graph has no !-vertices, and then describe how we plan to extend it to allow arbitrary !-graphs as the target.

The algorithm maintains a partial graph monomorphism from the pattern graph to the (concrete) target graph. It terminates when this is a complete morphism, and is successful if it is a local isomorphism.

The overall approach is to extend the graph morphism to the match surface (those parts of the pattern graph not contained in any unfixed !-boxes), then choose a top-level !-vertex and try killing it or expanding it. In the expansion branch, the match surface will have increased, so we repeat the graph morphism extension step, before trying another !-vertex. In the killing branch, we proceed straight to choosing another !-vertex. At each choice that could affect the final match, the algorithm branches, so as to explore all options. If it fails to extend the graph morphism to the match surface at any point, that branch terminates with failure.

Circles are matched first in each round of extending the graph morphism. Circle matching involves taking each unmatched circle in U_C and matching it to an arbitrary unmatched circle of the same type in T, and removing the vertex from T. Since circles are indistinguishable under rewriting (being wire-homeomorphic to each other), no branching needs to occur. Doing this first therefore reduces the amount of work duplicated across branches of the algorithm.

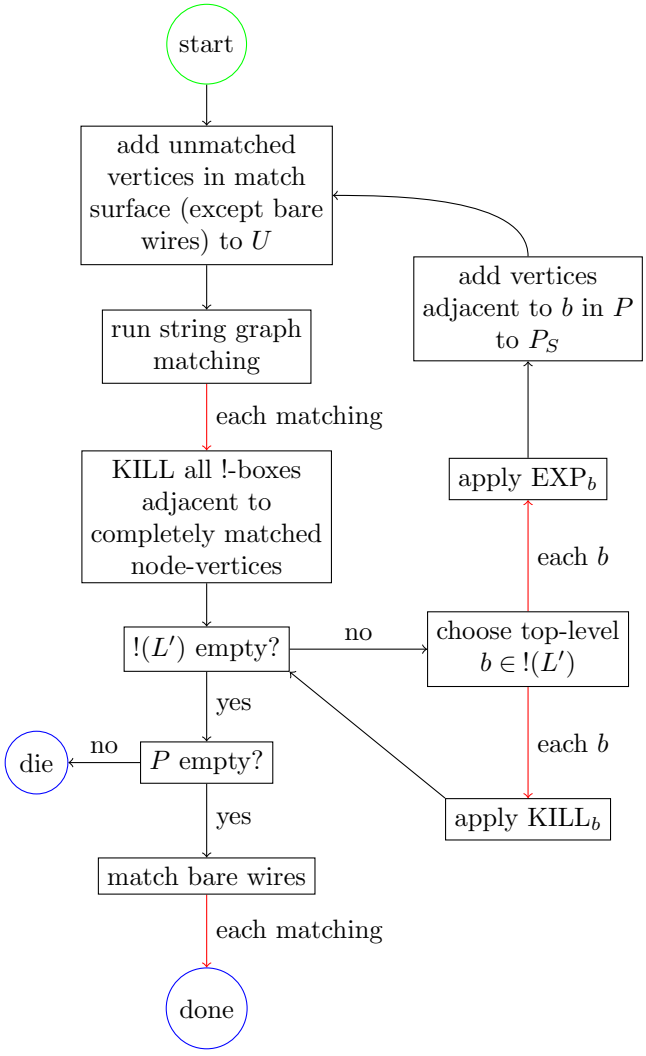


Figure 7.1: !-graph matching onto string graphs

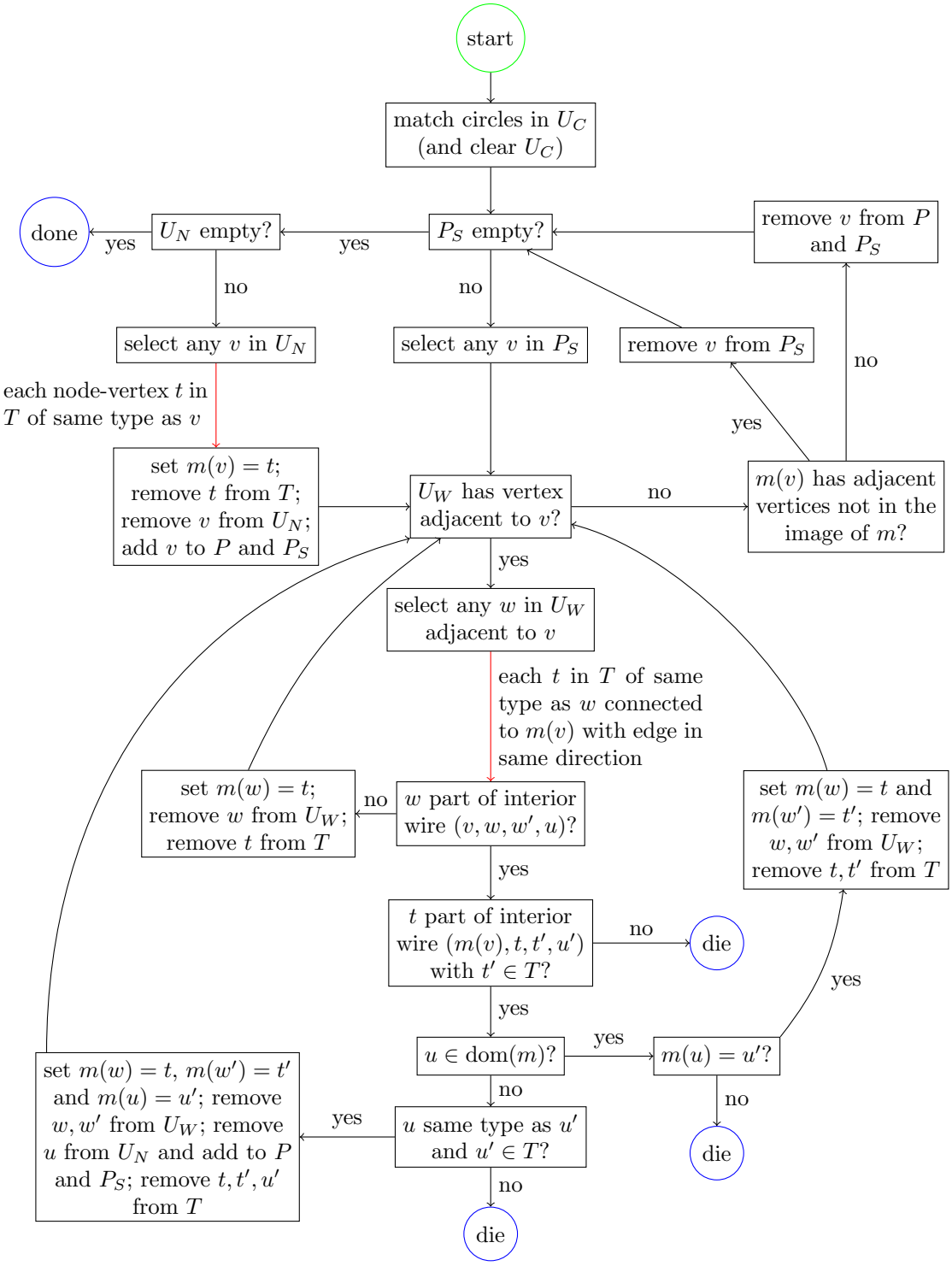
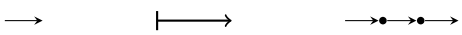


Figure 7.2: String graph matching subroutine

Bare wires are left until the very end of the matching process, after all the instantiation has been done. This is because (as noted in the discussion on wild !-boxes) bare wire matching does little to constrain the possible instantiations and can generate a lot of branches. For each bare wire $s \xrightarrow{e} t$ in the match surface, the bare wire matching step branches for each unmatched edge e' in G connected to a wire-vertex of the same type as the bare wire. It then replaces e' with three edges and two wire-vertices (of the appropriate type):



and matches $s \xrightarrow{e} t$ to the new wire-vertices and the edge between them.

To deal with wire homeomorphism, the algorithm starts with a variant of the normalisation scheme from section 3.6.2: every interior wire has two wire-vertices; bare wires have only the input and output vertices; circles have a single wire-vertex; and input and output wires have only the input or output vertex in the pattern graph but an extra wire-vertex in the target graph. This ensures that a wire that can accept a match from a bare wire (ie: any wire other than an interior wire already matched by an interior wire or a circle matched by a circle) always has at least one unmatched edge when the bare wire matching step is reached.

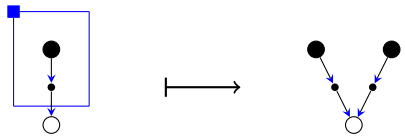
During a run of the algorithm to match a !-graph L onto a string graph G, Quantomatic maintains the following state:

- L : a normalised (as described above) !-graph
- L' : an instance of L
- S : a partial instantiation of L, resulting in L'
- G : a normalised (as described above) string graph
- $m : V_{L'} \to V_G$: a partial injective function describing the matching so far
- $U \subseteq V_{L'}$: a set of unmatched vertices of L'
- $P \subseteq N(L')$: a set of partially-matched node-vertices
- $P_S \subseteq P$: a set of scheduled partially-matched node-vertices
- $T \subseteq V_G$: a set of vertices that can be matched onto

Note that we do not maintain a map of edges; this is because !-graphs are simple, so if a function from vertices of one graph to vertices of another extends to a graph morphism, it does so uniquely.

U is partitioned into U_C, containing wire-vertices in circles (of which there can only be one for each circle), U_W, containing other wire-vertices, and U_N, containing node-vertices.

A note on terminology: a vertex is *matched* if it is in either the domain or image of m. If n is a matched node-vertex of L', we call it (and its image $m(n)$) *partially matched* if some of the neighbourhood of $m(n)$ is not in the image of m. Otherwise, it is *completely matched*. The same terminology applies to $m(n)$. For example, consider the following incomplete matching:



n has been matched to a vertex in the target graph, but we will need to expand the !-box twice before we are done with n. This is an example of a partially matched node-vertex.

P is (a superset of) the set of partially matched vertices of L', and P_S is intended to contain those node-vertices of P that may be able to become completely matched due to the application of a !-box operation.

We split the algorithm description into two parts. Figure 7.1 contains the part of the algorithm that explores the instantiation space, and figure 7.2 has the part that extends the match morphism. This latter part can be used in a simpler wrapper to match string graphs onto string graphs.

Red wires (with labels starting "each") indicate a branching point. A branch is created for each possibility. If there are no possibilities, the current branch dies. So, for example, if the algorithm takes a vertex from U_N and there are no possible matchings for it in the target graph, that branch will be killed off.

The initial match state has $L' = L$ and G normalised (in the manner described above), $S = m = U = P = P_S = \varnothing$, and T populated with all the vertices of G not in bare wires.

A proof of the correctness of this algorithm can be found in appendix A.

7.2.1 Matching Onto !-Graphs

In this section, we describe how the previous algorithm can be extended to match arbitrary !-graphs. This algorithm has been implemented for Quantomatic (albeit not in a released version yet).

Firstly, we need to maintain information about which !-vertices are fixed. Since !-vertices are only fixed at intermediate stages of a proof (when using Induct or during the matching process itself), we maintain the information about which !-vertices are fixed with which tags separately to the graphs. Thus we extend the match state (and algorithm inputs) with two partial maps, $F_L : !(L') \nrightarrow \mathcal{A}$ and $F_G : !(G) \nrightarrow \mathcal{A}$.

We also add two further sets to the match state to keep track of !-vertices:

- $B_S \subseteq \mathrm{dom}(F_L)$: a set of unmatched fixed !-vertices
- $B_C \subseteq !(L')$: a set of !-vertices that can be copied

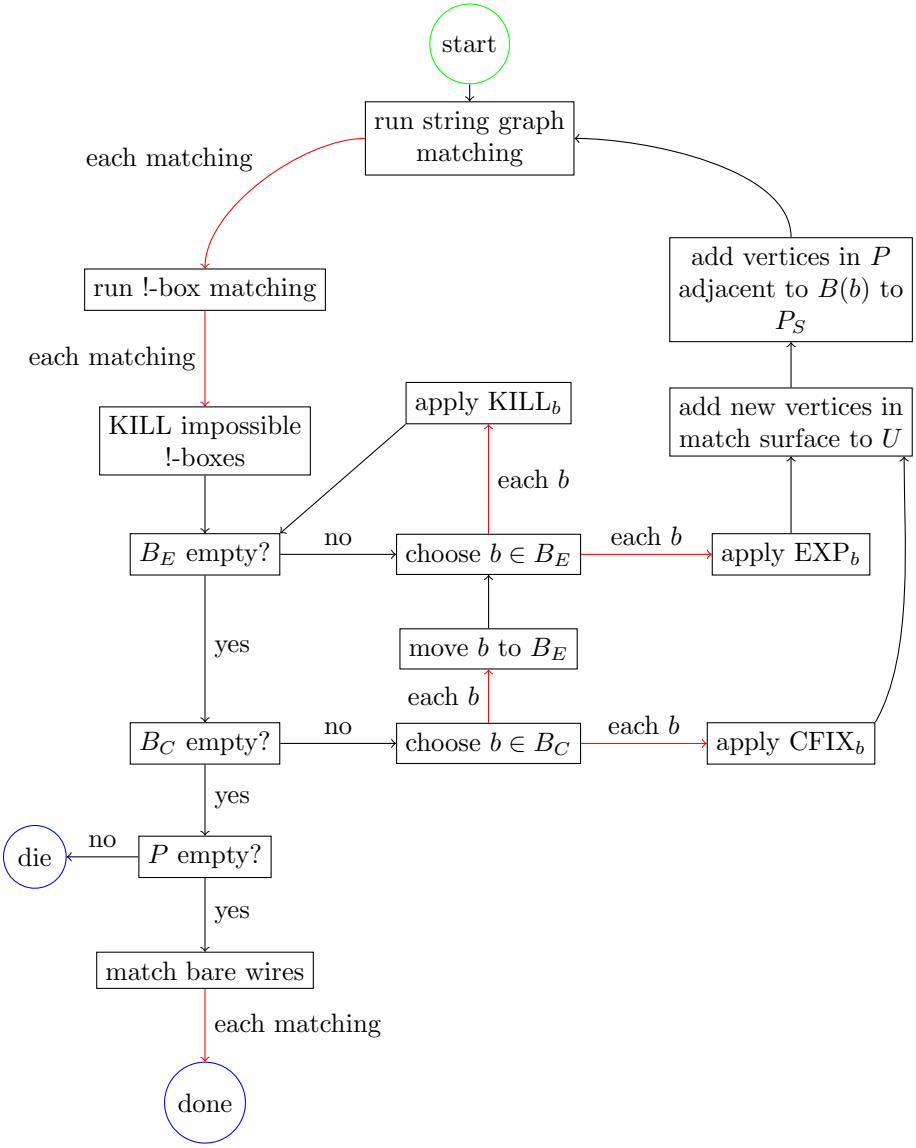


Figure 7.3: !-graph matching onto !-graphs

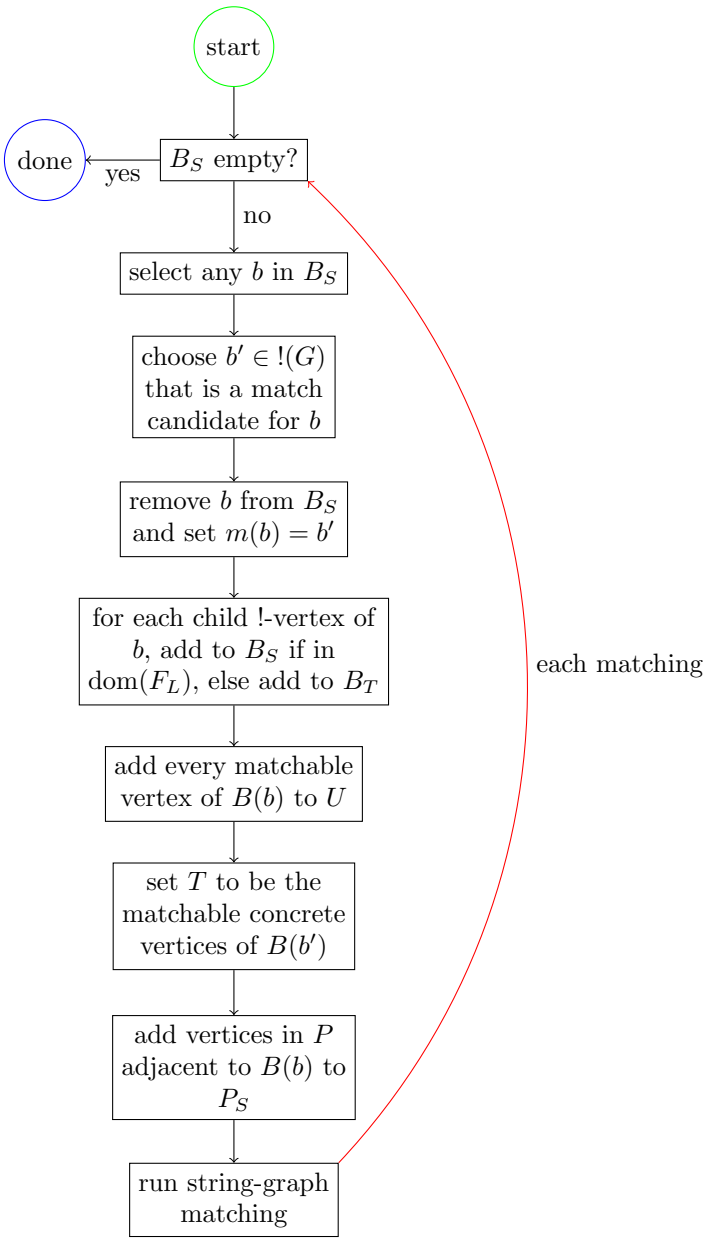


Figure 7.4: !-box matching subroutine

- $B_E \subseteq !(L')$: a set of !-vertices that can be expanded

The initial match state is mostly empty as before, with G and $L = L'$ again normalised. We also populate U with the vertices of L that are not in any !-box, and T with all the vertices of G that are not in any !-box. The new parts of the match state are populated as follows:

- $B_S = \mathrm{dom}(F_L)$
- $B_C = \{b \in !(L) \setminus B_S : \delta(b) = 0\}$
- $B_E = \varnothing$

The string graph matching subroutine is extended (via "hooks" in the implementation) to allow the matching to be constrained by !-box membership. In particular, we only allow a vertex $v \in V_L$ to match a vertex $v' \in V_G$ if $m[B_v] = B_{v'}$, where $B_v = \{b \in !(L) : v \in B(b)\}$ and $B_{v'} = \{b \in !(G) : v' \in B(b)\}$.

Also implicit in the algorithm description is that when we kill a !-vertex b, we always add any vertices in P that were adjacent to $B(b)$ to P_S, so that the next string graph matching run can remove them from P if they are now completely matched.

Additionally, one part of figure 7.4 refers to a "candidate for $m(b)$". If $b \in !(L) \setminus \mathrm{dom}(m)$, we say that $b' \in !(G) \setminus \mathrm{im}(m)$ is a *match candidate* for b if

- it shares the same parents (ie: $m[B^{\uparrow}(b) \setminus b] = B^{\uparrow}(b') \setminus b'$), and
- if b is fixed, b' is fixed with the same tag (ie: $(b \in \mathrm{dom}(F_L) \Rightarrow F_L(b) = F_G(b'))$)

The major difference between this algorithm and the previously-presented one is that CFIX_b^x is provided as an alternative choice to EXP_b and KILL_b when the algorithm deals with the next top-level !-vertex of L'.

Chapter 8

Conclusions and Further Work

In this dissertation, we have demonstrated how the string graph formalism for the diagrammatic languages of traced symmetric monoidal categories and compact closed categories can be extended to allow the finitary representation of infinite families of string graphs and string graph equations, and how these !-graphs and !-graph equations can be used in conjunction with double-pushout graph rewriting to do equational reasoning both with and on infinite families of morphisms in these categories. We have also presented some inference rules for a nascent logic of !-graphs.

We started by extending the language of monoidal signatures to allow for nodes with variable-arity edges, representing families of morphisms such as the spiders induced by commutative Frobenius algebras.

We then further extended string graphs with a language, internal to the graph, describing potentially infinite families of string graphs. We extended the notion of string graph equations to these !-graphs, constructing !-graph equations (and rewrite rules) to represent infinite familes of string graph equations (and rewrite rules).

We demonstrated how these !-graph rewrite rules can be used to both rewrite string graphs and to rewrite !-graphs, producing new !-graph rewrite rules. We showed that this rewriting is sound with respect to the interpretation of !-graphs and !-graph rewrite rules as families of string graphs and string graph rewrite rules.

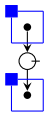
We described an equational logic of !-graphs implemented by !-graph rewriting and built on this with further inference rules, forming a logic of !-graphs that is again sound with respect to the interpretation of !-graphs as families of string graphs. This included a graphical analogue of induction, which we used to derive the spider law for commutative Frobenius algebras.

We rounded off by showing how !-graph rewriting can be implemented. We demonstrated that, providing a !-graph rewrite rule has no instantiations with wild !-boxes, it is possible to determine the finite set of instances of the rule that apply to a given !-graph, and gave an example of an algorithm to find matchings, with their associated instantiations, from the LHS of a !-graph rewrite rule to a string graph or !-graph.

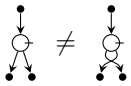
8.1 Further Work

The most obvious next step would be to remove the requirement that variable-arity edges of the same type commute by placing an order on the (variable-arity) edges of a node and using this order when determining the value of the elementary subgraph containing that node. This would allow us to have spiders for non-commutative Frobenius algebras.

A visual way to represent this could be to place a "starting mark" on a node, and count clockwise from there:

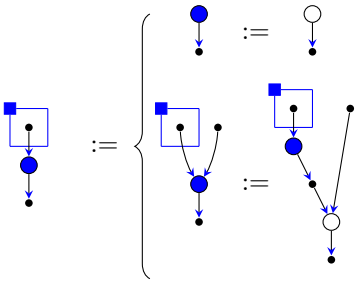


Then we would have that the following two graphs are not the same:

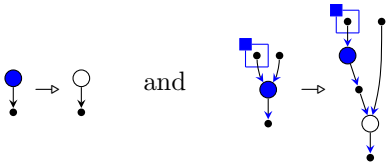


and, in particular, we would not allow a graph homomorphism from one to the other. The implications of this need working out in detail to ensure there are no unexpected side-effects; for example, there is a potential difference between expanding a !-box clockwise and anti-clockwise. It would allow more expressivity, though, and the commutative version presented in this thesis should be recoverable by placing constraints on the allowed valuations.

Another idea that could be investigated is "defined" generators. For example, suppose we have a commutative monoid $(A, \Yup, \Ydown)$. We could define a "spidered" version of this in the following manner:



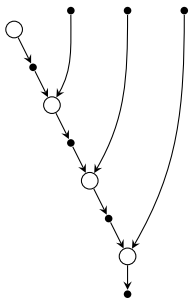
This essentially defines two !-graph rewrite rules



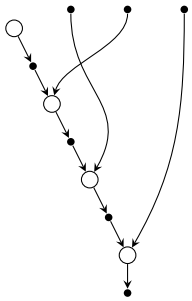
Some restrictions would need to be placed on such a rewrite system, such as termination and preservation of value, in order for it to be considered a definition, but this could be a powerful tool for proving theorems. Appendix B uses it to prove the spider law for a commutative Frobenius algebra.

The above definition has a major drawback, though, in that it implicitly requires that the monoid is commutative. For example, suppose we have the graph

This can be expanded to

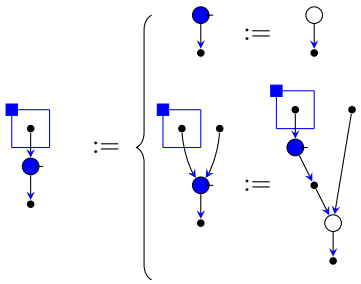


using the rewrite system implied by the definition, but it can also be expanded to



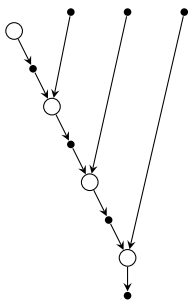
which, in the absense of commutativity and associativity laws, could have a different value. This is because all the incoming edges of the variable-arity multiplication have the same type, and so we have a choice of matches when rewriting.

However, if we combine this idea of definitions with the non-commutative generators already mentioned, we can make the definition



which will produce a unique expansion for each possible starting graph. In particular, it forces the incoming edges to this node to be expanded in clockwise order, so

has the unique expansion



There are undoubtably more inference rules to be discovered. For example, a variant on BOX(G) (section 6.3) that adds a fresh !-vertex b to G but only adds edges to itself and to vertices in $U(G)$ is likely to give rise to a similar inference rule to Box, but this has yet to be proved sound.

Kissinger introduced a notion of critical pairs for graphs in [22]. The rewriting of !-graphs laid out in this thesis could potentially be used, in combination with Kissinger's definition of critical pairs, to compute confluent extensions to rewrite systems of !-graphs using a Knuth-Bendix completion algorithm[19].

Of course, one of the most important steps to take is to actually implement the ideas presented in this thesis. In particular, we are planning to implement and prove the correctness of the algorithm in section 7.2.1. This algorithm could no doubt be improved, however, to make it more efficient. For example, many graphs involving commutative Frobneius algebras have a lot of internal symmetry, which results in multiple matches being found that all produce the same rewrite; it would be desirable to eliminate this duplication, preferably at the matching stage.[1]

The implementation of the ideas in this thesis is work that is currently underway in the Quantomatic project[25], which aims to be a useful proof assistant for string diagrams in compact closed and traced symmetric monoidal categories. This encompasses QuantoCosy, a tool for generating rules from models (as set out in [24]), QuantoDerive, a tool for constructing and checking graphical proofs, and QuantoTactic, an Isabelle[31] tactic implementation using Quantomatic's graph-matching engine to direct proofs and checking their correctness in Isabelle.

[1] Some initial work on this was done by Matvey Soloviev when the theory of !-graphs and string graphs was still in flux, but nothing was ever published; this could be revisited now that there is a secure foundation.

Appendix A

Correctness of Quantomatic's Matching

We prove the correctness of the algorithm presented in section 7.2 for matching !-graphs onto string graphs.

Recall the match state:

- L : the pattern graph (normalised)
- L' : an instance of L
- G : the target graph (normalised)
- S : a partial instantiation of L, resulting in L'
- $m : V_{L'} \to V_G$: a partial injective function describing the matching so far
- $U \subseteq V_{L'}$: a set of unmatched vertices of L'
- $P \subseteq N(L')$: a set of partially-matched node-vertices
- $P_S \subseteq P$: a set of schedules partially-matched node-vertices
- $T \subseteq V_G$: a set of vertices that can be matched onto

Throughout the entire procedure, in addition to the invariants implicit in the typing of the components of the match state, we maintain the following global invariants:

(1) m is injective and respects vertex types

(2) For any $v, w \in \mathrm{dom}(m)$, if there is an edge from v to w in L', then there is an edge from $m(v)$ to $m(w)$ in G

(3) $P \subseteq \mathrm{dom}(m)$ and for any node-vertex $v \in \mathrm{dom}(m) \setminus P$, all wire-vertices adjacent to $m(v)$ in G are in the image of m

Given that !-graphs are simple, invariants (1) and (2) ensure that when m is total, it uniquely extends to a !-graph monomorphism $\hat{m} : L' \to G$. Invariant (3) ensures that, if P is empty, $U(\hat{m})$ will be a local isomorphism. Thus $\hat{m}$ will be a matching of L' onto G under S.

A.1 The String Graph Matching Subroutine

The algorithm is shown in figure A.1.

A.1.1 Preconditions/Invariants

In addition to the global invariants, the string graph matching subroutine has the following requirements on the match state. What is more, these are maintained throughout the procedure.

(i) L' and G are both normalised

(ii) $U \cap \mathrm{dom}(m) = \varnothing$ (ie: the vertices in the unmatched set are not part of the existing matching)

(iii) Each vertex in U_W is adjacent to something in $U_N \cup P_S$ (ensures we can always reach wire vertices in U_W by starting from a node-vertex in U_N or P_S)

(iv) If $v \in P$ and $N_G(m(v)) \subseteq \mathrm{im}(m)$, $v \in P_S$ (everything that is completely matched and in P is also in P_S)

(v) If $v \in U_W$ and w is a wire-vertex adjacent to v in L', $w \in U_W$ (U_W contains complete wires)

(vi) If $v \in \mathrm{dom}(m)$ is a wire-vertex, then any vertices it is adjacent to are also in $\mathrm{dom}(m)$

(vii) If $v \in T$ is a wire-vertex, then any vertices it is adjacent to are in either T or $\mathrm{im}(m)$

(viii) $T \cap \mathrm{im}(m) = \varnothing$

A.1.2 Postconditions

These are true for the match state that results from each successful branch:

(I) $U = P_S = \varnothing$ (every vertex marked for matching was handled)

(II) $\mathrm{dom}(m)$ is the union of the initial states of U and $\mathrm{dom}(m)$ (we have matched exactly what we were asked to)

(III) $\mathrm{im}(m)$ contains only vertices that were in either the initial state of $\mathrm{im}(m)$ or the initial state of T (matches were only made against vertices in T)

(IV) P is exactly the set of vertices in $\mathrm{dom}(m)$ whose image is adjacent to a wire-vertex not in $\mathrm{im}(m)$ (P contains exactly the partially-matched vertices)

(V) L, L', G and S are identical to their starting states

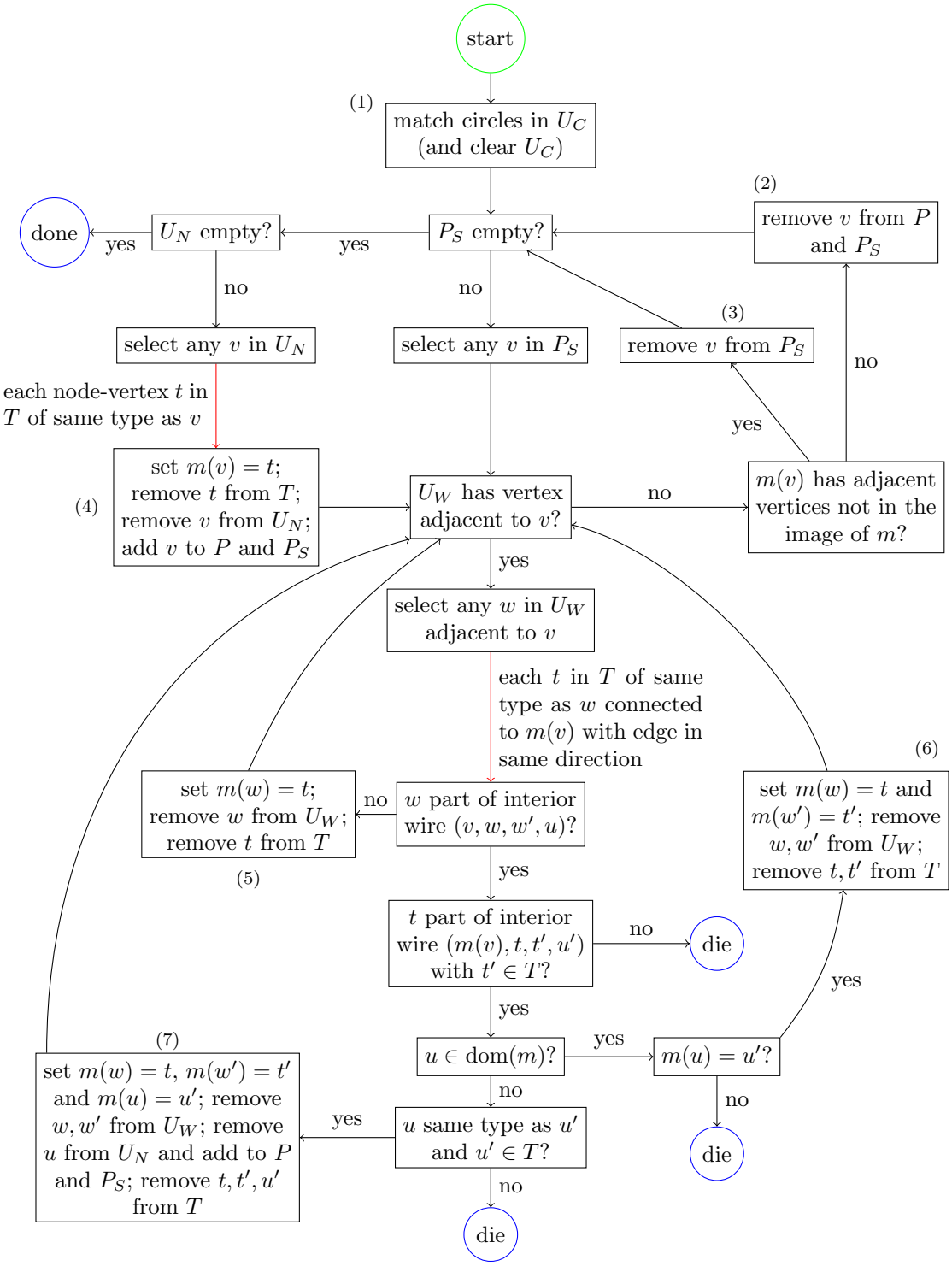


Figure A.1: String Graph Matching Subroutine

Postcondition (II) (in conjunction with the global invariants) states that the algorithm has completed the requested work. Postcondition (I) just states that we left the match state "clean"; it still satisfies all the preconditions and repeating the algorithm without altering the state further would do nothing. Postcondition (IV) says that P is the smallest set that satisfies global invariant (3).

A.1.3 Termination

For termination of the inner loop exploring the wires incident to a node-vertex v, we consider the variant $n = |U_W \cap N_{L'}(v)|$, where $N_{L'}(v)$ is the neighbourhood of v in L'. In each of these branches, n decreases by either 1 or 2, and nothing is ever added to U_W (and $N_{L'}(v)$ is fixed). So this loop must terminate.

The loop working through P_S has two branches: when it is not empty and when it is not. Either may add more things to P_S. We will take as variant $m = 2|U_N| + |P_S|$. When this is 0, the algorithm will terminate. Otherwise, if P_S is empty, U_N will have one vertex removed and added to P_S, decreasing m by one. If P_S is not empty, one vertex will be removed from P_S, decreasing m by one. Every vertex that is added to P_S in this loop is taken from U_N, further decreasing m. As a result, m will decrease by at least one with every iteration of the loop.

A.1.4 Correctness

There are seven steps of the algorithm that modify the match state; we will show that each of them satisfies the following properties, under the assumption that the preconditions (section A.1.1), global invariants and match state type constraints hold at the start of the step (note that when we say that a vertex is *taken from* a set, we mean that it was in the set at the start of the step, and is removed from the set by the end of the step):

(a) L, L', G and S are unmodified

(b) If U, T or m are modified, the only modifications are to take vertices from U and extend m to map those vertices to ones of the same type taken from T

(c) If a wire-vertex is added to $\mathrm{dom}(m)$, then so is the entire wire, and for any pair of vertices u, v in the wire with an edge from u to v, there is an edge from $m(u)$ to $m(v)$ (at the end of the step) in G

(d) Any vertex taken from U_N is added to both P and P_S, and these are the only vertices added to either P or P_S

(e) If P or P_S are modified, the only modifications are the one from (d), removing a partially matched vertex with no adjacent vertices in U_W from P_S or removing a completely matched vertex with no adjacent vertices in U_W from P and P_S

Given that these properties hold, we can show that all steps of the algorithm preserve the match state type constraints, the global invariants and the preconditions.

We can see that (a) holds for all seven steps by inspection. It is clear that the other three properties hold for steps (1)-(3), again by inspection. Only steps (5)-(7) need extra justification. We first note that the selection of w demands that, since L is in normal form, it is part of an interior wire with two wire-vertices or is the input on an input wire or the output on an output wire (and the sole wire-vertex on the wire in both cases).

For step (5), the only non-obvious property is (c). However, by what we have already noted, since w is not on an interior wire it must be an input or output. Since w and v are the only vertices on the wire and v is already in $\mathrm{dom}(m)$, and the edge between v and w is the same direction as the edge between $m(v)$ and t, property (c) holds for (5).

In step (6), it is clear that t and t' are in T and w is in U_W. Precondition (v) ensures that w' is in U_W. Since L and G are !-graphs, w' must have the same type as w, which has the same type as t, which has the same type as t. So property (b) clearly holds. Property (c) can be seen to hold by noting that the edge between u and w is in the same direction as the edge between $m(u)$ and t, and all the edges in a wire must be in the same direction. Properties (d) and (e) trivially hold, as U_N, P and P_S are not modified.

For step (7), we note that precondition (iii) (together with global invariant (3) and the fact that $P_S \subseteq P$) requires that, since $w' \in U_W$ and $u \notin \mathrm{im}(m)$, u must be in U_N. Then properties (b) and (c) follow in the same manner as for (6), and properties (d) and (e) clearly hold.

We now show that these properties mean that the match state type constraints, global invariants and preconditions are all maintained.

Type constraints The typing constraints for U and T are trivially maintained as (b) demands they are never added to. Likewise, the constraints on L, L', G and S are maintained as (a) means they are never changed. Properties (d) and (e) ensure that P is only ever added to from U_N and that anything added to P_S is also added to P, and anything removed from P is also removed from P_S, so those constraints hold. Finally, the constraint on m holds by property (b).

Global Invariants Injectivity of m is preserved as property (b) demands that only vertices drawn from T may be used for images of additions to m, and precondition (viii) gives us that these must not already be in the image of m. The rest of global invariant (1) follows from the same property.

Suppose there is a an edge in L from a vertex v to another vertex w, and both are in $\mathrm{dom}(m)$ at the end of a step, but one was not at the start of the step. We know they cannot both be node-vertices, and so one or both are wire-vertices. Since at least one was not in the image of m at the start of the step, none of the wire-vertices can have been by precondition (vi). Then property (c) ensures that there is an edge from $m(v)$ to $m(w)$. So global invariant (2) is preserved.

Global invariant (3) is preserved when vertices are removed from P due to property (e) and when vertices are added to m due to property (b).

Preconditions/Invariants

(i) L' and G are both in normal form

By property (a), which ensures L' and G are never changed.

(ii) $U \cap \mathrm{dom}(m) = \varnothing$

Property (b) requires that every vertex added to $\mathrm{dom}(m)$ is removed from U.

(iii) Each vertex in U_W is adjacent to something in $U_N \cup P_S$

Given that property (b) prevents anything being added to U_W, this precondition can only be broken by removing something from U_N or P_S. Property (d) ensures that anything removed from U_N is added to P_S, and property (e) ensures that anything removed from U_N has no adjacent vertices in U_W.

(iv) If $v \in P$ and $N_G(m(v)) \subseteq \mathrm{im}(m)$, $v \in P_S$

Property (e) ensures that when we add anything to P, we also add it to P_S, and we only remove completely matched vertices from P_S.

(v) If $v \in U_W$ and w is a wire-vertex adjacent to v in L', $w \in U_W$

Property (c) (in conjunction with property (b)) ensures that only entire wires are removed from U_W at once (and nothing is ever added to U_W).

(vi) If $v \in \mathrm{dom}(m)$ is a wire-vertex, then any vertices it is adjacent to are also in $\mathrm{dom}(m)$

This is preserved due to property (c).

(vii) If $v \in T$ is a wire-vertex, then any vertices it is adjacent to are in either T or $\mathrm{im}(m)$

This is maintained by property (b), which ensures that anything removed from T is added to $\mathrm{im}(m)$ and, due to precondition (ii), also ensures that nothing is ever removed from $\mathrm{im}(m)$.

(viii) $T \cap \mathrm{im}(m) = \varnothing$

This is preserved by property (b), which ensures that nothing is ever added to T, and anything added to $\mathrm{im}(m)$ is removed from T.

Postconditions

(I) $U = P_S = \varnothing$

The conditions that lead to the "done" step ensure that U_N and P_S are empty. Then precondition (iii) means that U_W must be empty. U_C is empty at the end because it is empty when step (1) is complete, and property (b) ensures it is never added to.

(II) $\mathrm{dom}(m)$ is the union of the initial states of U and $\mathrm{dom}(m)$

Property (b) ensures that everything removed from U is added to m, that nothing else is added to m and that nothing is removed from m. Combining this with precondition (ii) gives us that no entry of m is every overwritten, and so postcondition (I) provides us with what we need.

(III) $\mathrm{im}(m)$ contains only vertices that were in either the initial state of $\mathrm{im}(m)$ or the initial state of T

By property (b).

(IV) P is exactly the set of vertices in $\mathrm{dom}(m)$ whose image is adjacent to a vertex not in $\mathrm{im}(m)$

Global invariant (3) gives us most of what we need. We just need that if v is completely matched, it is not in P. But this is guaranteed by invariant (iv) and the fact that $P_S = \varnothing$.

(V) L, L', G and S are identical to their starting states

This follows immediately from property (a).

Completeness

Let f be a matching of the full subgraph of L' given by the vertices $U \cup \mathrm{dom}(m)$ into G that restricts to m on the vertices in $\mathrm{dom}(m)$. Then we need to show that, after completion of the algorithm, one of the match states produced will agree with f on vertices, up to a choice of circles. Note that, by the argument at the start of section A, this will then uniquely extend to f.

Since f is a matching with U_C in its domain (and $U_C \cap \mathrm{dom}(m) = \varnothing$ and f restricts to m), there must be at least $|U_C|$ unmatched circles in G, and so the step where we match circles must succeed.

Invariant: the vertex map part of f restricts to m. m is extended in two places (after circle-matching). Each of these follows a branching point. We need to show that in both of these places, there is a branch that will maintain this invariant. We also need to show that none of the "die" branches are taken.

When adding something from U_N, it is clear that $f(v)$ is a valid matching for v, and so this is a branch that will maintain the invariant.

When adding something from U_W, the wire starting $f(w)$ must be a valid matching for the wire starting with w, as the entire wire must be in the domain of f: if w is an input or output, this is

trivial. Otherwise, there must be a wire-vertex w' adjacent to w, and a node-vertex n adjacent to w'. Then w' is in U_W, and hence in dom(f), by precondition (v), and then n must be in either P_S or U_N, by precondition (iii), and so in dom(m) or U_N, and hence n is in dom(f). Once we have matched w against $f(w)$, the only possible matching for w' is $f(w')$, and similarly n must match against $f(n)$. If n is already in dom(m), we know it maps to $f(m)$, so the matchings must agree. So this branch maintains the invariant (and does not die).

Thus, regardless of how vertices are chosen from P_S, U_N or U_W, there must be a trace that returns a match where m agrees with f on vertices.

A.2 The Outer Loop

The algorithm is shown in figure A.2. The steps that "apply" a !-box operation implicitly apply the operation to L' and record it in S.

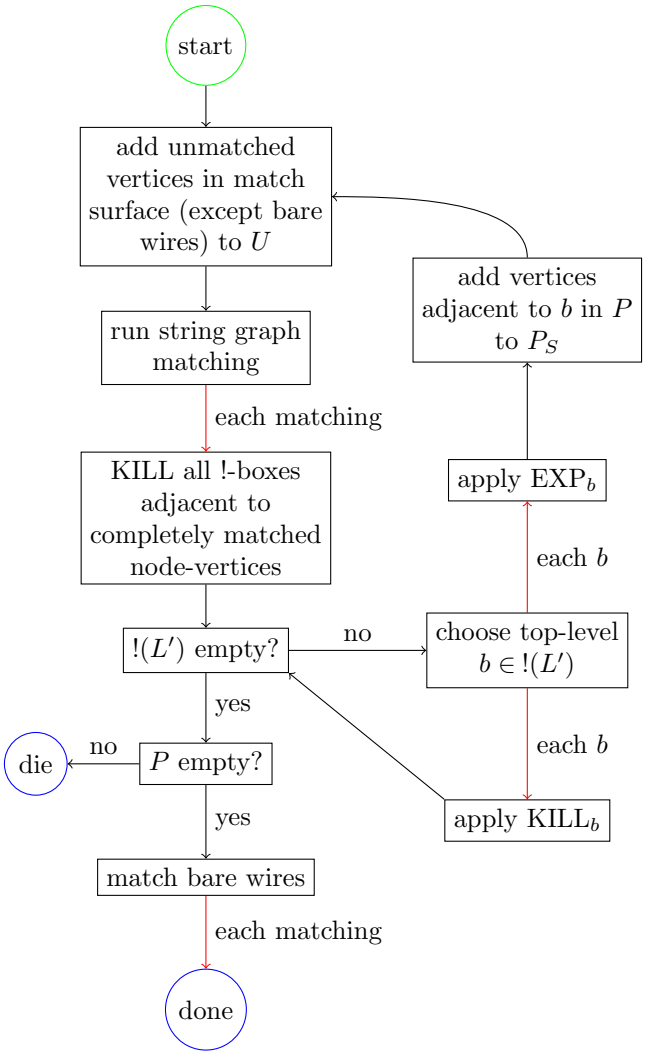


Figure A.2: !-graph matching onto string graphs

Note that the global invariants are trivially satisfied at the start of the algorithm, since m and P are empty. They are maintained thereafter as only the string graph matching subroutine and the bare wire matching routines alter m or P, and the !-box operations (which are the only things to alter L') can only add edges where either the source or the target is not in $\mathrm{dom}(m)$.

A.2.1 Termination

The algorithm terminates, on the assumption that no sequence of !-box operations will ever produce a wild !-box, by the following argument.

Each step terminates, and there is only one loop. As a variant, we take

$$n = |V_G| + |V_L \setminus \mathrm{dom}(m)| - |\mathrm{dom}(m)|$$

We calculate the variant at the point where we consider terminating the loop, after the string graph matching step. If the previous iteration involved killing one or more !-boxes, $V_L \setminus \mathrm{dom}(m)$ will decrease (by the combined size of the !-boxes, less bare wires — this amount is positive due to the assumption about wild !-boxes). If the previous iteration involved expanding a !-box, $|\mathrm{dom}(m)|$ will increase (by the size of the !-box, excluding any bare wires). $V_L \setminus \mathrm{dom}(m)$ is clearly bounded below by 0, and $|\mathrm{dom}(m)|$ is bounded above by $|V_G|$, by global invariant (1), so n is bounded below by 0 and decreases with every iteration. Hence the algorithm terminates.

Note that Quantomatic's implementation ensures termination for all pattern graphs by killing any wild !-boxes as soon as they are produced.

A.2.2 String Graph Matching Preconditions

We need to show that the preconditions in section A.1.1 are satisfied whenever the string graph matching step is run.

(i) L' and G are both normalised

We normalise the graphs at the start. We then do not alter G until the bare wire matching step, at which point the string graph matching routine can no longer be called. Because !-boxes are open subgraphs, operations on them always remove or copy an entire wire at once, which ensures that L' remains normalised throughout.

(ii) $U \cap \mathrm{dom}(m) = \varnothing$

We only ever add unmatched vertices to U.

(iii) Each vertex in U_W is adjacent to something in $U_N \cup P_S$

This follows from openness of !-boxes, which (together with normalisation) ensures that any wire-vertex in the match surface that is not on a circle or bare wire must be adjacent to a

node-vertex in the match surface. The first time string graph matching happens, all such node-vertices are put in U_N.

Since all non-bare-wire, non-circle vertices in the match surface are added to U_W each time, and so added to $\mathrm{dom}(m)$ by the string graph matching step, on subsequent iterations of string graph matching any wire-vertex v in U_W must have arisen from the expansion of a !-box (since this is the only way the match surface can be extended). If v is adjacent to a node-vertex that came from the !-box expansion, that node-vertex will be added to U_N. Otherwise, it must be adjacent to a node-vertex in $\mathrm{dom}(m)$, and hence in P (by the postconditions of the string graph matching step). Then this node-vertex (being adjacent to the expanded !-box) will be added to P_S.

(iv) If $v \in P$ and $N_G(m(v)) \subseteq \mathrm{im}(m)$, $v \in P_S$ (everything that is completely matched and in P is also in P_S)

The first time we run the string graph matching step, P is empty. After each string graph matching step, the postconditions ensure that there is no completely matched vertex in P. As the rest of the algorithm does not add anything to P or alter m, this is maintained throughout.

(v) If $v \in U_W$ and w is a wire-vertex adjacent to v in L', $w \in U_W$.

This is guaranteed by openness of !-boxes: we only ever add entire wires to U_W.

(vi) If $v \in \mathrm{dom}(m)$ is a wire-vertex, then any vertices it is adjacent to are also in $\mathrm{dom}(m)$

This is guaranteed by the postcondition (II) in combination with precondition (v) and the fact that m is only altered by the string graph matching step.

(vii) If $v \in T$ is a wire-vertex, then any vertices it is adjacent to are in either T or $\mathrm{im}(m)$

T and m are only modified by the string graph matching subroutine, which maintains this invariant. Initially, m is empty, causing it to hold trivially.

(viii) $T \cap \mathrm{im}(m) = \varnothing$

As above.

A.2.3 Correctness

At the end of the algorithm, everything that was ever in the match surface except for bare wires has been added to m by the string graph matching step (postcondition (II), together with the fact that we always add everything unmatched in the match surface to U_W, U_C and U_N, except bare wires). We have no more !-boxes, so the entire graph is part of the match surface. P is also empty. We matched bare wires at the last stage, so m is in fact total on L. So, by the argument at the start of section A, m is a valid match from the final state of L to G. Since we have only ever performed

valid !-box operations on L, this is a valid instance of L, and hence we have found a valid matching from L to G.

A.2.4 Completeness

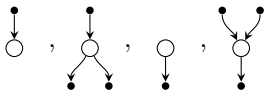
Providing no instance of L contains a wild !-box (see section 7.1), completeness follows from the fact that any concrete instantiation has an equivalent in expansion-normal form (section 6.2.2).

Appendix B

The Spider Law

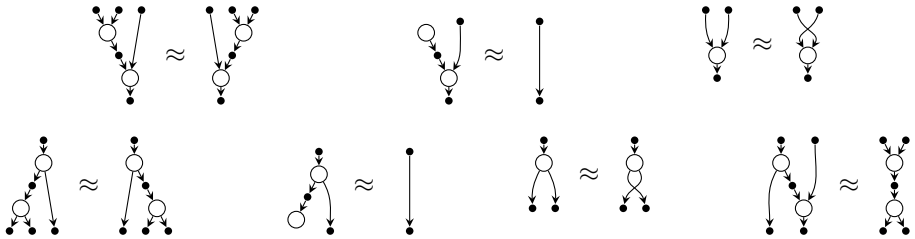
The following is joint work with Aleks Kissinger. It is work in progress, as we are still working out the details of the definition style used in this section and intend to extend this to non-commutative generators, as mentioned in section 8.1.

Suppose we have a commutative Frobenius algebra in string graph form. So we have the following generators (node-vertices of the typegraph, with their adjacent edges)

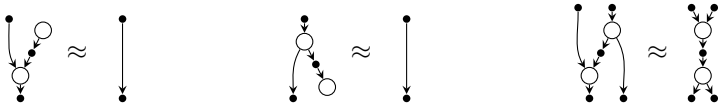


where all the edges are fixed-arity, and all the wire-vertices are of the same type. Note that, in the sequel, we will implicitly identify distinguished fixed-arity edges by the position in our presentation of the graph. This is unambiguous with these particular generators, as we can simply count clockwise from the single output of the multiplication node, or the single input of the comultiplication node.

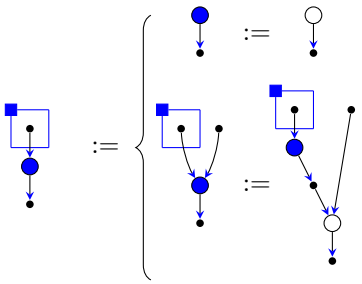
We also have the following equations, where we identify inputs and outputs by position:



It is trivial to use the commutativity laws to derive



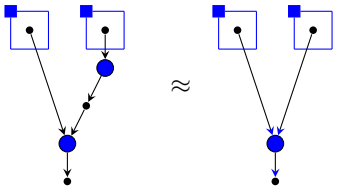
Suppose we now define a !-boxed version of the multiplication operation in the following manner:



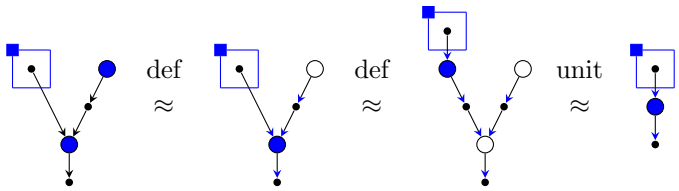
Essentially, we have added a new generator with a variable-arity input and a fixed-arity output, together with rules that allow us to rewrite any instance of it (with any number of incoming edges) into a graph containing only our original set of generators. As noted in section 8.1, this definition assumes associativity and commutativity in order to be consistent.

We now show that a partial form of the spider law holds for variable-arity multiplications.

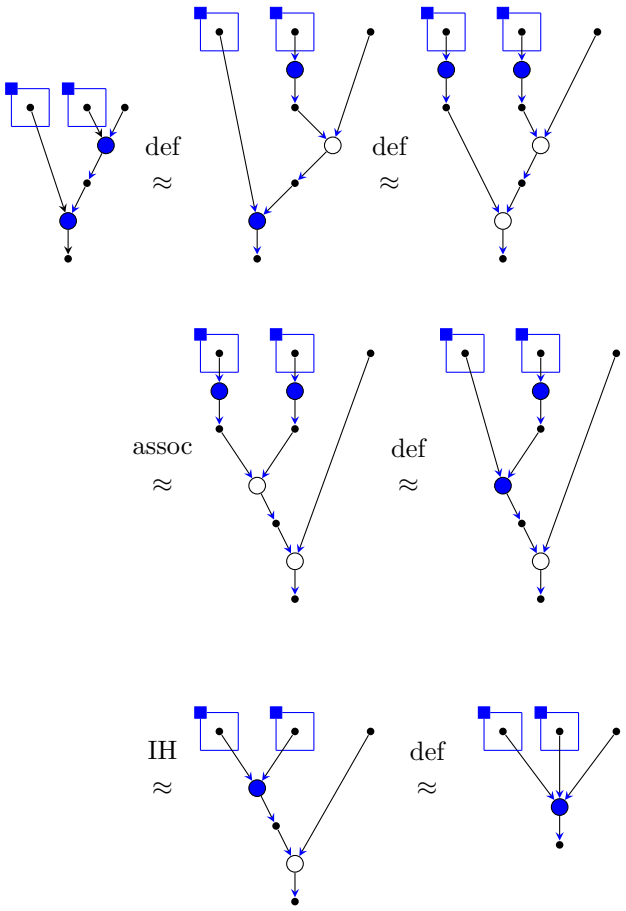
Lemma B.0.1.



Proof. We proceed by !-box induction. The base case:

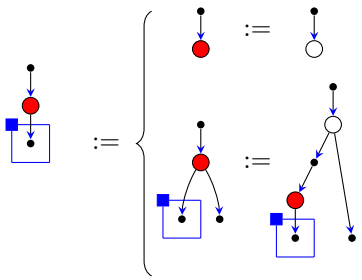


and the step case:



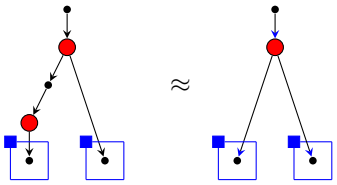
□

We define a variable-arity comultiplication in a similar manner:

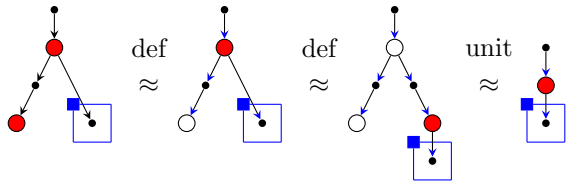


and prove a similar result about it:

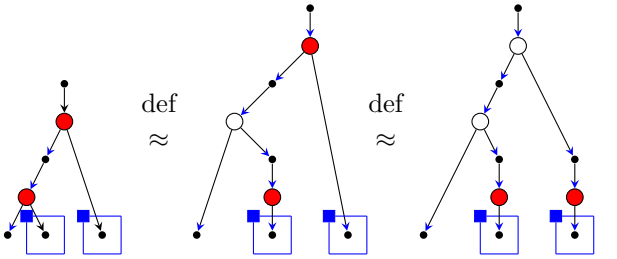
Lemma B.0.2.

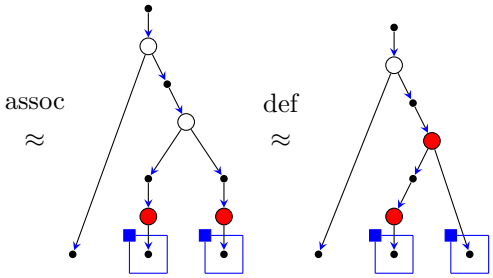


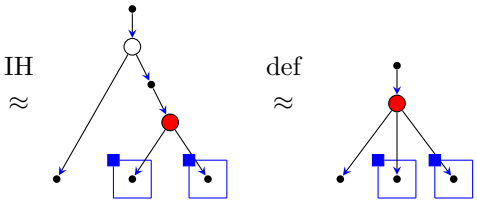
Proof. We proceed by !-box induction. The base case:



and the step case:

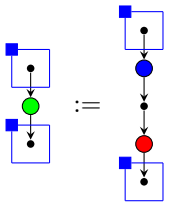






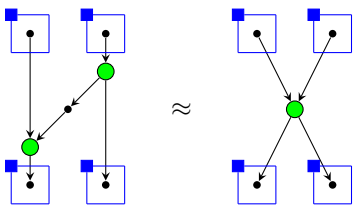
□

Now we are ready to define our spider:

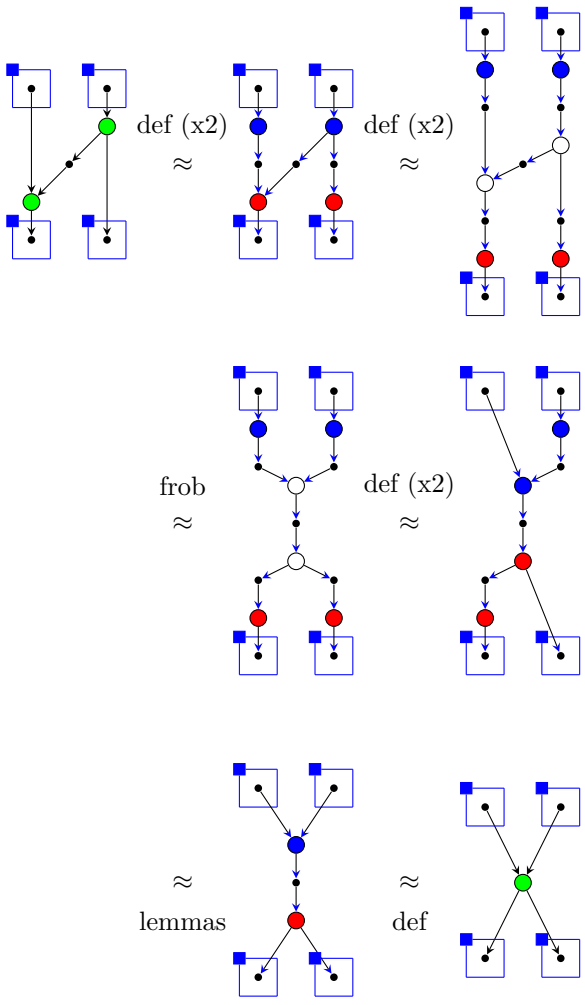


And the spider law:

Theorem B.0.3.



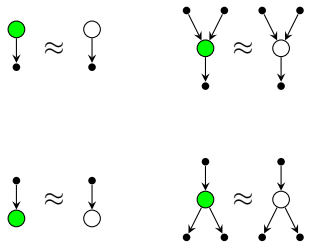
Proof.



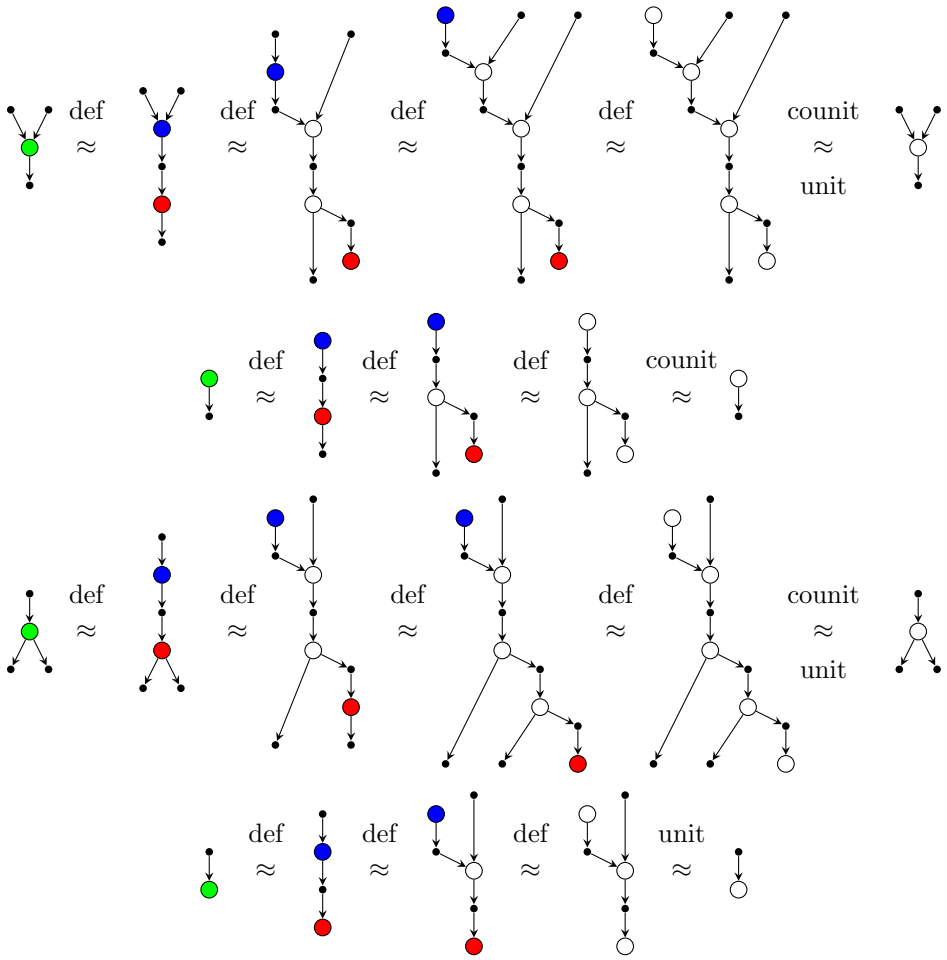
□

Of course, we need to ensure that our defined spider coincides with the original Frobenius algebra operations:

Theorem B.0.4.



Proof.


def
≈
def
≈
def
≈
def
≈
counit
≈
unit
def
≈
def
≈
def
≈
def
≈
counit
≈
def
≈
def
≈
def
≈
def
≈
counit
≈
unit
def
≈
def
≈
def
≈
unit
≈

□

Bibliography

[1] S. Abramsky and B. Coecke. A categorical semantics of quantum protocols. In *Proceedings of the 19th Annual IEEE Symposium on Logic in Computer Science (LICS)*, pages 415–425. IEEE, February 2004.

[2] F. Akinniyi, A. C. Wong, and D. Stacey. A new algorithm for graph monomorphism based on the projections of the product graph. *IEEE Transactions on Systems, Man, and Cybernetics*, 16(5):740–751, September 1986.

[3] F. Baader and T. Nipkow. *Term Rewriting and All That*. Cambridge University Press, 1998.

[4] I. Boneva, A. Rensink, M. E. Kurbán, and J. Bauer. Graph abstraction and abstract graph transformation. Technical report, University of Twente, July 2007.

[5] B. Coecke and R. Duncan. Interacting quantum observables: categorical algebra and diagrammatics. *New Journal of Physics*, 13(4):043016, April 2011.

[6] B. Coecke and A. Kissinger. The compositional structure of multipartite quantum entanglement. In *Proceedings of the 37th International Colloquium on Automata, Languages and Programming (ICALP)*, volume 6199 of *Lecture Notes in Computer Science*, pages 297–308. Springer Berlin Heidelberg, February 2010.

[7] B. Coecke, A. Kissinger, A. Merry, and S. Roy. The GHZ/W-calculus contains rational arithmetic. In *Proceedings of the CSR 2010 Workshop on High Productivity Computations*, volume 52 of *Electronic Proceedings in Theoretical Computer Science*, pages 34–48, March 2011.

[8] B. Coecke and D. Pavlovic. Quantum measurements without sums. In G. Chen, L. Kauffman, and S. J. Lomonaco, editors, *The Mathematics of Quantum Computation and Technology*, pages 559–596. Taylor and Francis, August 2007.

[9] L. Dixon and R. Duncan. Graphical reasoning in compact closed categories for quantum computation. *Annals of Mathematics and Artificial Intelligence*, 56(1):23–42, July 2009.

[10] L. Dixon, R. Duncan, and A. Kissinger. Open graphs and computational reasoning. In *Proceedings of the 6th on Developments in Computational Models (DCM)*, volume 26 of *Electronic Proceedings in Theoretical Computer Science*, pages 169–180, June 2010.

[11] L. Dixon and A. Kissinger. Open-graphs and monoidal theories. *Mathematical Structures in Computer Science*, 23(2):308–359, February 2013.

[12] R. Duncan and M. Lucas. Verifying the Steane code with Quantomatic. In *Proceedings of the 10th International Workshop on Quantum Physics and Logic (QPL)*, 2013.

[13] R. Duncan and S. Perdrix. Rewriting measurement-based quantum computations with generalised flow. In *Proceedings of the 37th International Colloquium on Automata, Languages and Programming (ICALP)*, volume 6199 of *Lecture Notes in Computer Science*, pages 285–296. Springer Berlin Heidelberg, 2010.

[14] H. Ehrig and H.-J. Kreowski. Church-Rosser theorems leading to parallel and canonical derivations for graph-grammars. Technical report, TU Berlin, 1975.

[15] H. Ehrig, M. Pfender, and H. J. Schneider. Graph-grammars: an algebraic approach. In *Conference Record of the 14th Annual Symposium on Switching and Automata Theory (SWAT)*, pages 167–180. IEEE, October 1973.

[16] J.-Y. Girard. Linear logic. *Theoretical Computer Science*, 50(1):1–101, 1987.

[17] D. Gries and F. B. Schneider. *A Logical Approach to Discrete Mathematics*. Springer Verlag, 1993.

[18] M. Hasegawa, M. Hofmann, and G. Plotkin. Finite dimensional vector spaces are complete for traced symmetric monoidal categories. In A. Avron, N. Dershowitz, and A. Rabinovich, editors, *Pillars of Computer Science*, Lecture Notes in Computer Science, pages 367–385. Springer Berlin Heidelberg, 2008.

[19] G. Huet. A complete proof of correctness of the Knuth-Bendix completion algorithm. *Journal of Computer and System Sciences*, 23(1):11–21, August 1981.

[20] A. Joyal and R. Street. The geometry of tensor calculus I. *Advances in Mathematics*, 88(1):55–112, July 1991.

[21] A. Joyal, R. Street, and D. Verity. Traced monoidal categories. *Mathematical Proceedings of the Cambridge Philosophical Society*, 119(03):447–468, April 1996.

[22] A. Kissinger. *Graph rewrite systems for classical structures in †-symmetric monoidal categories*. MSc Thesis, University of Oxford, 2008.

[23] A. Kissinger. *Pictures of Processes*. DPhil Thesis, University of Oxford, 2011.

[24] A. Kissinger. Synthesising graphical theories. In *Proceedings of the 1st Workshop on Automated Theory eXploration (ATX)*, February 2012.

[25] A. Kissinger, A. Merry, L. Dixon, R. Duncan, M. Soloviev, and B. Frot. Quantomatic, https://sites.google.com/site/quantomatic/.

[26] A. Kissinger, A. Merry, and M. Soloviev. Pattern graph rewrite systems. In *Proceedings of the 8th International Workshop on Developments in Computational Models (DCM)*, Electronic Proceedings in Theoretical Computer Science, 2012.

[27] S. Lack and P. Sobociński. Adhesive and quasiadhesive categories. *Theoretical Informatics and Applications*, 39(3):511–545, 2005.

[28] A. Lang and B. Coecke. Trichromatic open digraphs for understanding qubits. In *Proceedings of the 8th International Workshop on Quantum Physics and Logic (QPL)*, volume 95 of *Electronic Proceedings in Theoretical Computer Science*, pages 193–209, October 2011.

[29] S. Mac Lane. *Categories for the Working Mathematician*. Springer Verlag, 2nd edition, 1998.

[30] U. Nickel, J. Niere, and A. Zündorf. The FuJaBa environment. In *Proceedings of the 22nd International Conference on Software Engineering (ICSE)*, pages 742–745, 2000.

[31] T. Nipkow, L. C. Paulson, and M. Wenzel. *Isabelle/HOL: A Proof Assistant for Higher-Order Logic*. Lecture Notes in Computer Science. Springer Berlin Heidelberg, 2002.

[32] R. Penrose. Applications of negative dimensional tensors. In D. J. A. Welsh, editor, *Combinatorial Mathematics and its Applications*, pages 221–244. Academic Press, New York, New York, USA, 1971.

[33] J. L. Pfaltz and A. Rosenfeld. Web grammars. In *Proceedings of the 1st International Joint Conference on Artificial Intelligence*, pages 609–619. Morgan Kaufmann Publishers Inc., 1969.

[34] A. Rensink. Nested quantification in graph transformation rules. In *Proceedings of the 3rd International Conference on Graph Transformations (ICGT)*, volume 4178 of *Lecture Notes in Computer Science*, pages 1–13. Springer Berlin Heidelberg, September 2006.

[35] A. Rensink and J.-H. Kuperus. Repotting the geraniums: on nested graph transformation rules. *Electronic Communications of the EASST*, 18, 2009.

[36] H. J. Schneider. Chomsky-Systeme für partielle Ordnungen. Technical report, Universität Erlangen, 1970.

[37] P. Selinger. A survey of graphical languages for monoidal categories. In B. Coecke, editor, *New Structures for Physics*, volume 813, pages 289–355. Springer Berlin Heidelberg, August 2009.

[38] F. Stallmann. *A model-driven approach to multi-agent system design.* PhD Thesis, University of Paderborn, 2008.

[39] G. Taentzer. *Parallel and distributed graph transformation: formal description and application to communication-based systems.* PhD Thesis, TU Berlin, 1996.

[40] J. R. Ullmann. An algorithm for subgraph isomorphism. *Journal of the ACM*, 23(1):31–42, January 1976.

[41] P. M. van den Broek. Algebraic graph rewriting using a single pushout. In *Proceedings of the International Joint Conference on Theory and Practice of Software Development (TAPSOFT)*, volume 493 of *Lecture Notes in Computer Science*, pages 90–102. Springer Berlin Heidelberg, 1991.